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# *Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations.**

By N. J. Lennes.

## § 1. *Introduction.*

This paper contains a body of definitions and theorems relating to simple curves, limit-curves, etc., which, it is hoped, will be of general usefulness in a considerable range of non-metrical investigations in analysis situs and related subjects. It had its origin in an attempt to prove certain theorems concerning the existence of solutions in the calculus of variations. Indeed, the definition of an arc given in § 4 is merely an enumeration of those properties of a certain set of limit-points of a sequence of arcs which appear when one attempts to prove directly that they constitute a continuous arc.

It is apparent that in a geometry possessing linear order and continuity curves and limit curves exist entirely independently of metric properties. Hence the discussion so far as it relates to these is carried out on non-metric hypothesis. Schoenflies testifies to the desirability of this procedure in the following words (after quoting Cantor's definition of "Zusammenhang," which is stated in terms of equality of segments): "Wenn nun auch der Abstand zweier Punkte für die hier vorliegenden Untersuchungen einen axiomatischen geometrischen Grundbegriff bildet, so scheint es mir doch zweckmässig, rein mengentheoretische Definitionen überall da zu bevorzugen, wo es möglich ist, . . . . ." In spite of this explicit expression of preference for non-metric treatment "wo es möglich ist," Schoenflies uses metric hypothesis in the proof of practically every important theorem dealing with curves and the regions defined by them.

The argumentation in various parts of the paper requires a considerable body of theorems on simple finite and infinite polygons. Consequently § 2 is

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* Read before the Chicago Section of the American Mathematical Society at its December meeting, 1905. Changes made since then are entirely unimportant.

devoted to polygons. A number of theorems on the finite polygon are proved by the writer in a paper in this Journal.* Schoenflies proves the main theorem of § 3; viz., that an infinite continuous simple polygon separates the plane into two connected sets.† His treatment, however, makes use of full metric properties as well as the axiom of parallels, and also makes use, without proof, of the theorem for the finite case as stated by Hilbert and Veblen.‡ The latter theorem, however, had been proved earlier by Schoenflies from metric hypothesis.§ While the axiomatic basis for this treatment of the infinite polygon is thus considerably weaker than that used by Schoenflies, it is believed that the treatment is shorter than his, while at the same time less is left to be supplied by the reader.

Section 3 deals with approach to limit-points. To prove that there exists a sequence of points on a line approaching a given point as a limit-point, a mild form of continuity is used (see p. 305). This axiom may be regarded as the projective geometry analogue of the Archimedean axiom of metric geometry. So far as known to the writer the axiom in this form is new. The existence of a sequence of sets of points closing down *uniformly* upon a given closed set of points follows immediately without further axioms. The existence of such sequences is fundamental in the discussions that follow, and it is believed they will be generally useful in work on non-metric analysis situs. Indeed, it seems that metric properties have been brought in precisely at that point in the argument where such sequences of sets are here used, and that even by those who have avoided the use of metric properties most consciously and most successfully. Compare for instance Veblen's "Curves in Non-Metrical Analysis Situs" || with p. 312 of this paper. The given closed set upon which these sequences of sets of points close down is identical with the *generalized inner limiting set* of Young.¶ The *uniformity* of approach, however, is peculiar to this paper and it is this *uniformity* which is of importance in the argument.

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* Lennes: "Theorems on the Finite Polygon and Polyhedron," Vol. XXXIII (1911), pp. 37–62. For other non-metrical proofs of some of these theorems, see O. Veblen, *Transactions of the American Mathematical Society*, Vol. V (1904), p. 343, and Hans Hahn, *Monatshefte für Mathematik und Physik*, Vol. XIX, pp. 289–303.

† Schoenflies: "Beiträge zur Mengenlehre," I, *Mathematische Annalen*, Vol. LVIII, pp. 195–234.

‡ Hilbert: "Grundlagen der Geometrie" (2d edition, p. 6); and Veblen, *loco citato*, p. 365.

§ *Gött. Nachr.*, 1902, pp. 185–192.

|| *Transactions of the American Mathematical Society*, Vol. VI, p. 83.

¶ W. H. Young and Grace Chisholm Young: "The Theory of Sets of Points," *Cambridge University Press*, p. 69 *et seq.*

The construction used in § 3 to obtain a sequence approaching a given point as a limit is that given by Von Staudt* in his proof of the fundamental theorem of projective geometry. The axiom used in this paper is of course weaker than the full continuity used by Klein † to validate the argument of Von Staudt.

Section 4 contains the definition of arc (or curve) and a proof that it is an arc of a Jordan curve when the definition of the latter is couched in non-metric terms. Various point-set definitions of arcs (or curves) have been given. Veblen ‡ defines "curve" in terms of "point" and "order" and proves that the result is a Jordan curve. However, metric properties are used at one step in showing that the curve is actually a Jordan curve, — a result obtained in this paper by means of uniformity of approach. Schoenflies § defines "curve" as a frontier or outer rim of a connected region having the property of being accessible (p. 312) at every point both from exterior and interior points. Thus the Schoenflies definition of closed continuous curve is analogous to the Dedekind "Schnitt" on the line.

Young ‖ defines a curved arc as "a plane set of points, dense nowhere in the plane, such that, given any norm $e$, and describing around each point of the set a region of span less than $e$, these regions generate a single region $Re$, whose span does not decrease indefinitely with $e$." In Young's treatment free use is made of metric properties.

In this paper an arc is defined (p. 308) as follows: "*A closed, bounded, connected set of points containing $A$ and $B$, $A \neq B$, which contains no proper connected subset containing $A$ and $B$, is a continuous arc whose end-points are $A$ and $B$.*" This definition seems to be very near the obvious intuitional meaning of the term "arc" or "curve." It has the two properties of "connectedness" and "thinness"; viz., an arc consists of "one piece" and is so "thin," everywhere, that removing any one point, other than an end-point, separates it into two parts.

In section 5 the frontier of a region is considered as a Jordan curve. A proof is given of the theorem of Schoenflies that any outer rim of a connected

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* Von Staudt: "Geometrie der Lage," p. 50.

† *Mathematische Annalen*, Vol. VI, p. 139.

‡ Veblen: "Curves in Non-metrical Analysis Situs," *Transactions of the American Mathematical Society*, Vol. VI, pp. 83–98.

§ Schoenflies, *loco citato*, p. 195.

‖ W. H. Young and Grace Chisholm Young, *loco citato*, p. 206. Also W. H. Young: *Quarterly Journal of Pure and Applied Mathematics*, Vol. XXVII, pp. 1–35.

region is a Jordan curve in case it is accessible at every point both from exterior and interior points. It is also shown that an outer rim every point of which is accessible from an exterior point separates the remaining points of the plane into two connected sets, and also that a rim may be accessible at every point from exterior points and fail to be accessible from interior points, and hence need not be a Jordan curve.

Section 6 is devoted to a simple non-metrical proof of the classical theorem that a Jordan curve separates a plane into two connected sets. Section 7 is concerned with limit-arcs of a set of arcs. It is shown that under a certain uniformity condition on the continuity of the set of arcs at least one limit-curve exists.

In section 8 the general theory of the paper is applied to the problem of proving the existence of minimizing curves in an important class of problems in the calculus of variation.

The argumentation is based specifically on the axioms of Professor Veblen.*

The undefined symbols of Veblen's axioms are *point* and *order*. He defines a line containing the points $A$ and $B$ as consisting of the points $A$ and $B$ together with all points $X$ which have one of the orders $XAB$, $AXB$ and $ABX$. The points $X$ such that the order $AXB$ exists constitute the segment $AB$. If the points $A$, $B$, $C$ are not collinear, the segments $AB$, $BC$, $CA$, together with the points $A$, $B$, $C$, form a triangle, and all points collinear with two points of a fixed triangle form a plane.

The following axioms are used:

AXIOM I.† *There exist at least three points.*

AXIOM II. *If the points $A$, $B$, $C$ are in the order $ABC$, they are in the order $CBA$.*

AXIOM III. *If the points $A$, $B$, $C$ are in the order $ABC$, they are not in the order $BCA$.*

AXIOM IV. *If the points $A$, $B$, $C$ are in the order $ABC$, then $A$ is distinct from $C$.*

AXIOM V. *If $A$ and $B$ are two distinct points, there exists a point $C$ such that $A$, $B$, $C$ are in the order $ABC$.*

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* Oswald Veblen: "A System of Axioms for Geometry," *Transactions of the American Mathematical Society*, Vol. V (1904), pp. 345–384.

† The Roman numeral indicates the number of the axiom in Veblen's set.

Axiom VI. *If the points $C$ and $D$ ($C \not\equiv D$) lie on the line $AB$, then $A$ lies on the line $CD$.*

Axiom VII. *If there exist three distinct points, there exist three points $A$, $B$, $C$ not in any of the orders $ABC$, $BCA$, or $CAB$.*

Axiom VIII (the Triangle Transversal Axiom). *If three distinct points $A$, $B$, $C$ do not lie in the same line, and $D$ and $E$ are points in the orders $BCD$ and $CEA$, then a point $F$ exists in the order $AFB$ such that $D$, $E$, $F$ lie on the same line.*

Axiom C (Axiom of Continuity).* *If all points of a line are divided into two sets such that no point of either set lies between points of the other, then there is one point on the line which does not lie between points of either set.*

The theorems of section 2 are proved by means of Axioms I–VIII. The other theorems are proved by means of Axioms I–VIII and C. We adopt Veblen's definition of segment, line, triangle and plane. The discussion throughout the paper is confined to the plane, and the axioms selected from Veblen's set are plane axioms.

## § 2. *The Simple Polygon.*

The main topic of section 2 is the infinite continuous polygon. Theorems on the finite polygon that are used in the argumentation are inserted for convenience of reference. These theorems were proved explicitly in a paper in the present volume of this Journal.† The references to that paper are by page numbers and the number of the proposition; as (28), p. 45. References to propositions in the present paper are by the numbers of the proposition and section only; as (1), § 2. In some cases where the proofs are entirely obvious no reference is made.

Definitions. *A set of points $[X]$ ‡ such that one of the orders $AXB$ and $ABX$ exists, together with the points $A$ and $B$, forms a "half-line" $AB$ (not a half-line $BA$). The half-line is said to proceed from $A$.*

*The points lying on two half-lines proceeding from the same point but not lying in the same line form an "angle."*

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* This form of the axiom of continuity is, in the presence of Axioms I–VIII, equivalent to Axiom XI of Veblen.

† Lennes: "Theorems on the Simple Polygon and Polyhedron," American Journal of Mathematics, Vol. XXXIII (1911), pp. 37–62.

‡ The symbol $[X]$ is used to denote a set any one of whose elements may be denoted by the symbol within the brackets or by this symbol with subscript or other identifying marks. The brackets [ ] are used when the set is not in any particular order. If the set is ordered, we write $\{ X \}$.

The symbols $\angle$ and $\Delta$ are used for angle and triangle in the usual manner.

*A point $P$ is an "interior point" of a set if there is a triangle $t$ of which $P$ is an interior point such that every interior point of $t$ (possibly except $P$) is a point of the set.*

*A set of points is "entirely open" if every one of its points is an interior point of the set.*

1. THEOREM. *Any line of a plane separates the remaining points of the plane into two entirely open sets such that a segment connecting two points of the same set contains no point of the line, while a segment connecting points of different sets contains a point of the line.*

(For proof see E. H. Moore: "On the Projective Axioms of Geometry," *Transactions of the American Mathematical Society*, Vol. III (1902), pp. 142–158, or Veblen, *loco citato.*)

2. THEOREM. *An angle (triangle) separates the remaining points of the plane in which it lies into two entirely open sets, an interior and an exterior, such that a segment connecting an interior and an exterior point contains one point of the angle (triangle), a segment connecting two interior points contains no point of the angle (triangle) and a segment connecting two exterior points and not containing a vertex contains two or no points of the angle (triangle).*

(For proof see same as preceding.)

DEFINITIONS. *The points lying on a set of segments $A_1A_2$, $A_2A_3, \ldots, A_{n-1}A_n$, together with the points $A_1, A_2, \ldots, A_n$ (called vertices), constitute a finite broken line.*

*The point $L$ is said to be an end-point or limit-point of the infinite set of segments $A_1A_2$, $A_2A_3, \ldots, A_nA_{n+1}, \ldots$, if for every triangle $t$ of which $L$ is an interior point there is a number $N$ (depending on the triangle $t$) such that for every $n > N$ the segment $A_nA_{n+1}$ lies entirely within the triangle. The set of segments form an "infinite broken line" connecting its end-points $A_1$ and $L$. If a point $C$ is connected with $L$ or $A_1$ by means of a finite or infinite broken line, then the two broken lines together form a broken line connecting $A_1$ and $C$ or $L$ and $C$. Such points as $A_1$, $L$, $C$ are vertices of the broken line. A segment including its end-points is a special case of a broken line.*

Hereafter the expression "broken line" will be used for both finite and infinite broken lines. The word "finite" or "infinite" will be used when we wish to specify particularly the one or the other.

*If no point of a broken line is common to two of its segments, a segment and a*

*vertex, or two vertices (except possibly the end-points), the broken line is a "simple" broken line.*

*If a simple broken line connects two points $A$ and $B$, and if these points are the same point, the broken line forms a "simple polygon." If the broken line is finite, the polygon is "finite"; and if the broken line is infinite, the polygon is "infinite." The segments of the broken line are the "sides" of the polygon, and the vertices of the broken line are the "vertices of the polygon."*

*If a vertex is a limit-point of an infinite sequence of segments, the polygon is said to be "infinite" at this vertex or to have a "limit-point" at this vertex.*

The word "polygon" will be used for both finite and infinite simple polygons.

*An entirely open set of points is said to be connected* (*see note, p.* 303) *if for any two points of the set there is a broken line connecting them which lies entirely in the set.*

3. Theorem. *If $A$ and $B$ are points of an entirely open connected set, then there is a finite broken line connecting them which lies entirely in the set.*

Proof. By hypothesis there is a broken line (finite or infinite) in the set connecting the points $A$ and $B$. Suppose the broken line is infinite and has just one limit-point $L$. Since $L$ lies within the set, there is a triangle $t$ containing $L$ as an interior point all of whose interior points are points of the set. If $A$ is exterior to $t$, trace the given broken line from $A$ to a point on $t$ and likewise from $B$ to a point on $t$. These two finite broken lines, together with a segment connecting end-points of them within $t$, form the required broken line connecting $A$ and $B$. Since the broken line has only a finite number of vertices which are limit-points, it follows that a repetition of this construction gives the requïred broken line for the general case.

Definitions. *A set of points $[P]$ is said to "separate" the remaining points of the plane into two sets if every broken line connecting a point in one set with a point in the other contains at least one point of $[P]$.**

*If a triangle $t_i$ is constructed about each vertex $L_i$ of a set of broken lines $[b]$, then the segments of $[t_i]$, together with those segments of $[b]$ which are partly or entirely exterior to every triangle of $[t_i]$, are called the "exposed" set of $[b]$ with respect to $[t_i]$.*

* The following is a more general definition of *separation* : "A set of points $[P]$ is said to separate a connected set $[R]$ if the points of $[R]$ not in $[P]$ do not form a connected set," the term *connected set* being used in the sense of § 3. See page 303.

4. THEOREM. *If [b] is a finite set of broken lines, the remaining points of the plane form an entirely open set.*

PROOF. Let $L_1, L_2, \ldots, L_n$, or $[L_i]$, be the limit-vertices of $[b]$ and $P$ any point not of $[b]$. By (17), p. 41, there is for each point $L_i$ a triangle $t_i$ of which $P$ is an exterior point. By the definition of "continuous broken line" there is only a finite number of exposed segments of $[b]$ with respect to $[t_i]$. Hence by (17), p. 41, there is a triangle $t$ of which $P$ is interior, and every exposed segment, together with its limit-points, exterior. Then, by (16), p. 41, there is no point of $[b]$ within $t$.

5. THEOREM. *If $[b]$ is a finite set of broken lines and $ABC$ any angle, $B$ not a limit-vertex of one of the broken lines, then there is a ray $BK$ within the angle $ABC$ which contains no vertex of $[b]$.*

PROOF. If there are $n$ limit-vertices of $[b]$ on or within $\angle ABC$, construct rays from $B$ within the angle forming $2n+1$ angles of which no two have an interior point in common (8), p. 40. Then there is at least one angle of this set such that there is no limit-vertex of $[b]$ on or within it. Hence, by (2), § 2, and the definition of broken line, there are only a finite number of vertices of $[b]$ within this angle; and hence, by (8), p. 40, the required ray $BK$ may be constructed within it.

6. THEOREM. *If $[b]$ is any finite set of broken lines and $ABC$ an angle such that there is no point of $[b]$ on the segments $AB$ and $BC$ or their end-points, then there is a point $C'$ on $BC$ such that there is no point of $[b]$ on or within the triangle $ABC'$.*

PROOF. About each limit-vertex $L_i$ of $[b]$ on or within $\angle ABC$ construct a triangle $t_i$ such that no point of the segments $AB$ and $BC$ or their end-points lies on or within a triangle $t_i$. Then there is only a finite number of exposed segments within $\angle ABC$, and hence, by (15), p. 41, there is a point $C'$ on $BC$ such that there is no point of an exposed segment on or within the triangle $ABC'$, and hence no point of $[b]$ on or within this triangle.

7. THEOREM. *If $t_1$ is a triangle enclosing a limit-vertex $L$ of a set of broken lines $[b]$, then there is a triangle $t_2$ also enclosing $L$, which lies entirely within $t_1$ and on which lies no vertex of $[b]$. An infinite broken line connecting a point $A_1$ exterior to $t_2$ with its only limit-vertex $L$ within $t_2$ meets $t_2$ in an odd number of points.*

PROOF. Let $ABC$ be the triangle $t_1$ enclosing $L$. Using (5), construct $CD$ and $CE$ so that no vertex of $[b]$ lies on these segments while $L$ is within the

angle $DCE$. Similarly construct $DF$ and $DG$ and then $FH$, thus obtaining the triangle $FGH$ which has the required properties. That an infinite broken line connecting an exterior point $A_1$ with its only limit-point $L$ within $t_2$ meets $t_2$ in an odd number of points is then an obvious corollary of (2) and the definition of continuous broken line.

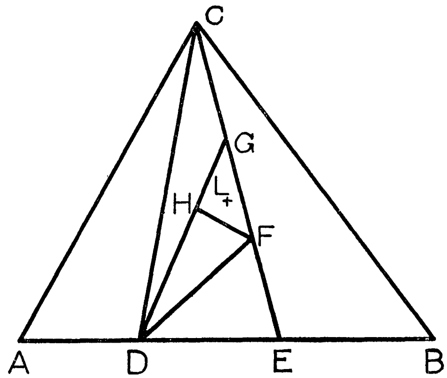


FIG. 1.

8. THEOREM. *If a line contains no vertex of a polygon, or if of an angle and a polygon neither contains a vertex of the other, then such line or angle contains an even number of points of the polygon, zero being an even number.*

PROOF. For the case when the polygon is finite, see (18), p. 42. In case it is infinite, proceed as follows: Let $l$ denote the given line and $L_1, L_2, \ldots, L_n$, or $[L_i]$, be the limit-vertices of the polygon, the notation being so arranged that the points are in that order on the polygon which is indicated by the subscripts. Let $[t_i]$ be a set of triangles such that $L_i$ lies within $t_i$ while every point of $l$ is exterior to every triangle of $[t_i]$ (see (1)). Consider the broken line $L_1 L_2$ which consists of two broken lines $A_1 A_2, A_2 A_3, \ldots, A_n A_{n+1}, \ldots, L_1$ and $A_1 A_2', A_2' A_3', \ldots, A_m' A_{m+1}', \ldots, L_2$. By definition (p. 292) there is an $N$ such that, for $n > N$, $A_n A_{n+1}$ lies within $t_1$, and an $M$ such that, for $m > M$, $A_m' A_{m+1}'$ lies within $t_2$. Then every point of the broken line $L_1 L_2$ which lies on $l$ is on the finite broken line $A_m' A_{m-1}', \ldots, A_{n-1} A_n$. As an immediate consequence of (1), this broken line contains an even or odd number of points on $l$ according as $A_n$ and $A_m'$ lie on the same or opposite sides of the line Since $L_1$ and $A_n$, and $L_2$ and $A_m'$ are on the same side respectively of $l$, it follows that the broken line $L_1 L_2$ contains an even or odd number of points of $l$ according as $L_1$ and $L_2$ are on the same or opposite sides of $l$. The theorem now follows

exactly as in the case of the finite polygon. The argument for the angle is identical with that given for the line, except that (2) is used instead of (1).

We now define as in the case of the finite polygon.

DEFINITION. *A point not on a polygon is an interior point of the polygon if a half-line proceeding from it and containing no vertex of polygon contains an odd number of points of the polygon. The point is exterior if such half-line contains an even number of points of the polygon.*

9. THEOREM. *If a broken line contains no point of a polygon, it is either entirely exterior or entirely interior.*

PROOF. For the case when the polygon is finite, see (19), p. 43. It remains to make the proof in case the polygon has one or more limit-vertices. We show first that a segment which does not meet the polygon is entirely interior or entirely exterior.

Let $A$ be any point of such segment $A_1A_2$ or its end-point $A_1$. Denote by $[L_i]$ the set of limit-vertices of the polygon. About each point $L_i$ construct a triangle $t_i$ of which the segment $A_1A_2$ is entirely exterior. Construct a half-line $AK$ not meeting a vertex of the polygon (5). Then by (6) there is a point $B$ on $AK$ such that there is no point of the polygon on or within the triangle $ABA_2$. Again, by (5) there is a ray $A_2H$ within $\angle AA_2B$ which contains no vertex of the polygon. Let the ray $A_2H$ meet the segment $AB$ in $R$ ((6), p. 39). Since the rays $AR$ and $A_2R$ contain no vertices of the polygon, and the segments $AR$ and $A_2R$ or their end-points contain no points of the polygon, it follows from the definition of exterior and interior points that the points on these segments, including their end-points, are all exterior or all interior; that is, $A$ and $A_2$ are both exterior or both interior. But $A$ is any point of the segment $A_1A_2$, or possibly $A_1$, and hence the points of this segment, including its end-points, are all interior or all exterior. It now follows immediately that any finite broken line which fails to meet the polygon is all interior or all exterior.

Consider now an infinite broken line $A_1A_2$, $A_2A_3$, $\ldots$, $A_nA_{n+1}$, $\ldots$ with a limit-vertex $L$ which does not meet the polygon. Then, by the preceding, the points of this broken line, except $L$, are all interior or all exterior. Since $L$ does not lie on the polygon, there is by (4) a triangle $t$ containing $L$ as an interior point within which there is no point of the polygon. Connect $L$ with some point $K$ of the broken line $A_1L$ within $t$. Then we have a finite broken line connecting $A_1$ and $L$, and hence $L$ is interior or exterior according as the remainder of the broken line is interior or exterior.

10. THEOREM. *If $P$ is a point of a side $A_1 A_2$ of a polygon, and if segments $PB$ and $PC$ lie on opposite sides of the line $A_1 A_2$ and contain no point of the polygon, then one segment is entirely exterior and the other entirely interior.*

PROOF. Through $P$ construct a line $l$ such that one ray $PK$ on it does not contain a vertex, (5). Let $B'$ and $C'$ be points on $l$ in the order $B'PC'$ such that $B$ and $B'$ lie on the same side of the line $A_1 A_2$, and such that there is no point of the polygon on $B'C'$ except the point $P$. Then, by definition, one of the points $B'$ and $C'$ is interior and the other is exterior. Since $B$ and $B'$ are on

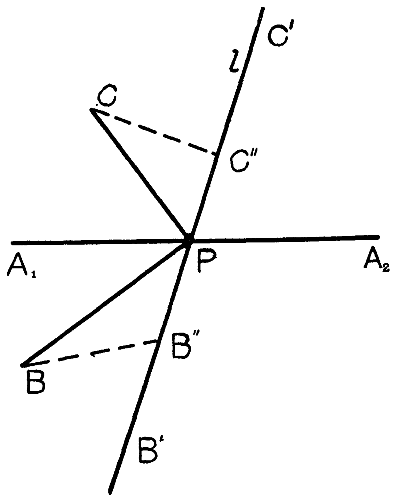


FIG. 2.

the same side of the line $A_1 A_2$, there is no point of the segment $A_1 A_2$ within the angle $BPB'$. Hence, by (6) there is a point $B''$ on $B'P$ such that there is no point of the polygon (except $P$) on or within the triangle $BPB''$. Hence, by (9) all points of the segment $BP$ are exterior or interior according as $B'$ is exterior or interior. In the same manner we show that the segment $CP$ is exterior or interior according as $C'$ is exterior or interior. Hence one of the segments $BP$ and $CP$ is entirely interior and the other entirely exterior.

It follows also from this argument that:

11. THEOREM. *If two segments $AB$ and $AC$ are both interior or both exterior and have the common end-point $A$ on a side of the polygon, then there is a broken line connecting $B$ and $C$ which does not meet the polygon.*

12. THEOREM. *If two points P and Q lie on the same or different sides of a polygon p, then there is a finite broken line connecting them which is entirely interior, and another which is entirely exterior.*

PROOF. Suppose $P$ and $Q$ lie on two consecutive sides $A_1 A_2$ and $A_2 A_3$ respectively. Then there is a point $H$ on the line $A_3 A_2$ in the order $A_3 A_2 H$ such that there is no point of $p$ on $A_2 H$. Then by (6) there is a point $M$ on $A_2 H$ such that there is no point of $p$ on $PM$. In the same manner we obtain a point $N$ on the line $PM$ in the order $PMN$ such that there is no point of $p$ on $QN$. Hence there is no point of $p$ on the broken line $PN$, $NQ$.

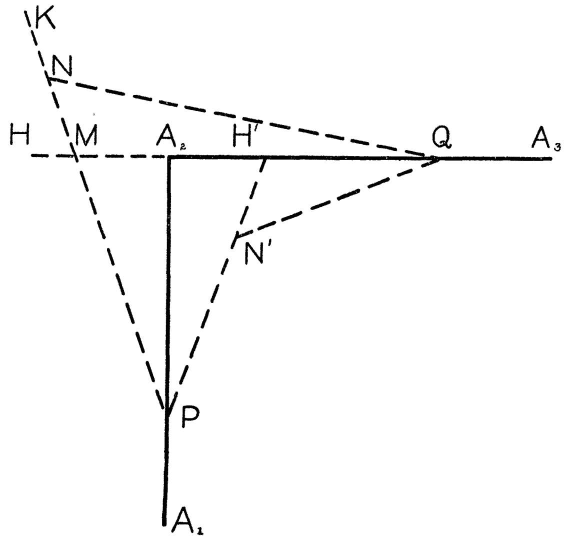


FIG. 3.

In the same manner we find $H'$ on the segment $QA_2$ and a point $N'$ on the segment $PH'$ such that there is no point of $p$ on the broken line $PN'$, $N'Q$. By (10) and (9) one of these broken lines is interior and the other exterior. By repeating this process and using (11) we now prove the theorem for the case where $P$ and $Q$ are connected by a finite broken line of the polygon.

We next consider the case where the points $P$ and $Q$ are connected by a broken line of $p$ which contains only one limit-vertex $L$. According to (7) enclose $L$ by a triangle on which lies no vertex of $p$ and no point of $p$ except of the broken line connecting $P$ and $Q$. Then by (7) each of the broken lines $PL$ and $QL$ meets $t$ in an odd number of points. Let $R_1 R_2, \ldots, R_n$ be the points in which it meets $PL$ as they appear in order on the triangle $t$. Then

there are two consecutive points, as $R_1$, $R_2$, on $t$ between which $QL$ meets $t$ in an odd number of points. Let these points in their order be $Q_1, Q_2, \ldots, Q_m$. Then by (10) one of the segments $R_1Q_1$ and $Q_mR_2$ is interior and the other is exterior. Suppose $R_1Q_1$ exterior. Then, by the finite case of the theorem and (11), there are finite broken lines connecting both $P$ and $Q$ with points on $R_1Q_1$ which are entirely exterior. These two broken lines, together with a segment of $R_1Q_1$, form a finite broken line connecting $P$ and $Q$ which is entirely exterior.

In the same manner, using the interior segment $Q_mR_2$, we obtain an interior broken line connecting $P$ and $Q$.

13. THEOREM. *A polygon separates the remaining points of the plane into two entirely open connected sets, one consisting of the interior points and the other of the exterior points of the polygon.*

PROOF. (a) By (9) a broken line connecting an interior and an exterior point meets the polygon.

(b) Any two interior points are connected by a broken line which does not meet the polygon. Let $M$ and $N$ be any two interior points. Connect these with points $P$ and $Q$ on the polygon by means of segments $MP$ and $NQ$ which contain no point of the polygon. Then, by (12), $M$ and $N$ may be connected. If $M$ and $N$ are both exterior points we proceed in the same manner.

DEFINITIONS. *The sets* $[O']$ *and* $[O'']$ *are complementary subsets of the set* $[O]$ *if* (a) *the sets* $[O']$ *and* $[O'']$ *have no element in common,* (b) *every element of either* $[O']$ *or* $[O'']$ *is an element of the set* $[O]$, (c) *every element of* $[O]$ *is an element of either* $[O']$ *or* $[O'']$.

*A set of points is bounded if there exists a polygon of which every point of the set is an interior point.*

*A point is an interior point of a set of polygons if it is an interior point of one polygon of the set.*

*A set of polygons is overlapping if any two complementary subsets have interior points in common.*

*Two points* $P_1$ *and* $P_2$ *are said to be mutually accessible with respect to a set of points* $[R]$ *if there exists a broken line connecting them but containing no point of* $[R]$ *except possibly* $P_1$ *or* $P_2$ *or both.*

14. THEOREM. *If two polygons are not identical and have interior points in common, then there are some points of one polygon within the other, and some points of one polygon exterior to the other.*

PROOF. Denote the polygons by $p_1$ and $p_2$. Since each polygon is simple, it follows that not all points of either lie on the other. Let $P$ be an interior point of both $p_1$ and $p_2$. If there are no points of $p_1$ within $p_2$, then all points of $p_2$ are accessible from $P$ with respect to $p_1$. Let $Q$ be a point of $p_2$, not of $p_1$. Then $Q$ lies within $p_1$, since it is accessible from the interior point $P$ and does not lie on the polygon itself. In the same manner we show that there are some points of one exterior to the other.

15. THEOREM. *If $[p]$ is a finite set of finite overlapping polygons, there is a finite polygon $p'$ all of whose points are points of $[p]$ such that all interior points of the set $[p]$ are interior points of $p'$.*

PROOF. On a line $l$ let $P$ be a point such that all intersections of $l$ with $[p]$ lie on the same side of $P$. Denote by $p'$ all points of $[p]$ accessible from $P$, and by $[I]$ all points not thus accessible. Then (a) no point of $p'$ lies within a polygon of $[p]$ and every segment of $p'$ is reached from $P$ from the exterior side of the polygon of $[p]$ on which it lies.

(b) All interior points of the set $[p]$ are points of $[I]$, since no such point can be reached from a point exterior to all polygons of $[p]$.

(c) Any two points, both interior, of the set $[p]$ are mutually accessible with respect to $p'$. Suppose the polygons of $[p]$ are ordered as $p_1, p_2, \ldots, p_n$ in such manner that $p_i$ and $p_{i+1}$ have interior points in common ($i = 1, \ldots, n-1$); then clearly any two interior points of $p_i$ and $p_{i+1}$ are mutually accessible, since one of these polygons contains points which lie within the other (14), and hence are not points of $p'$ (a).

(d) Let $I_1$ be any point of $[I]$ not an interior point of the set $[p]$. Connect $I$ with a point $Q$ on a segment of a polygon $p_1$ of $[p]$. Then $I_1Q$ is exterior to the polygon $p_1$, while $I_1$ is not accessible from $P$ with respect to $p'$. Hence, by (a), $Q$ is not a point of $p'$ whence $I_1$ can be joined to a point within $p_1$ without meeting $p'$. Hence $[I]$ is a connected set with respect to $p'$. Clearly the set of points not of $p'$ which are accessible from $P$ is a connected set. Hence $p'$ is a finite set of segments separating the remaining points of the plane into two connected sets. Clearly no subset of $p'$ does thus separate the plane, since removing a single point from $p'$ enables us to reach points of $[I]$ from $P$. Hence, by (27), p. 44, $p'$ is a simple finite polygon.

DEFINITION. *A point $L$ is a "limit-point" of a set of points $[P]$ if there are points of $[P]$ other than $L$ within every triangle of which $L$ is an interior point.*

16. THEOREM. *An exterior point of a polygon is not a limit-point of interior points, and an interior point is not a limit-point of exterior points.*

PROOF. This is a direct consequence of (4) and (14).

17. THEOREM. *A broken line which lies entirely within a polygon, except its end-points which lie on the polygon, forms with the polygon two polygons having no interior point in common such that all interior points of the first polygon lie on or within the two resulting polygons.*

PROOF. Denote the broken line by $P_1 P_2$. It is a consequence of the definition of polygon that two polygons are thus formed, the broken line $P_1 P_2$ being part of each polygon. Denote these two polygons by $p_1$ and $p_2$. Since every point of each polygon not of $P_1 P_2$ is accessible from some external point, it follows that neither polygon contains a point within the other, and hence by (14) they have no common interior point. That every interior point of the original polygon is on or within $p_1$ or $p_2$ is a direct consequence of the definition of interior points.

DEFINITION. *A broken line $b$ is said to cross a polygon $p$ once between two points $P_1$ and $P_2$ of $b$ if one of the two points, as $P_1$, is exterior and the other is interior, and if, following $b$ from $P_1$ to $P_2$, one is never led back from interior to exterior points. The polygon is also said to cross the broken line.*

*It will be noticed that some segments of $b$ and $p$ may coincide.*

18. THEOREM. *A broken line $AB$, finite or at most having the limit-vertices $A$ and $B$, crosses a polygon $p$ an odd number of times if $A$ is exterior and $B$ is interior and if $AB$ contains no limit-vertex of $p$.*

PROOF. This is an immediate consequence of (13).

19. THEOREM. *If $p_1$ is a finite polygon or infinite at most at the points $A$ and $B$, and if $p_2$ is a finite polygon of which $A$ is exterior and $B$ is interior, then $p_2$ contains a broken line which connects a point on one of the broken lines $AB$ of $p_1$ with a point of the other broken line $AB$ of $p_1$, and which lies entirely within $p_1$.*

PROOF. The polygon $p_1$ contains two broken lines $AB$ which we denote by $b_1$ and $b_2$. By (18) each of the broken lines crosses the polygon $p_2$ on odd number of times. The polygons $p_1$ and $p_2$ clearly have interior points in common, since points of the one lie within the other, and hence by (14) there are points of $p_2$ exterior to $p_1$. Let $Q$ be any such point. Suppose the theorem not true. Following the polygon $p_2$ from the point $Q$ we can meet $b_2$ only after having crossed $b_1$ an even number of times (zero being an even number), for otherwise just before meeting $b_2$ we should trace a broken line within $p_1$ such as we suppose

does not exist. Similarly we can not meet $b_1$ again until we have first crossed $b_2$ on even number of times. Continuing in this way, remembering that $p_2$ is a finite polygon, we show that $p_2$ crosses $b_1$ and $b_2$ an even number of times, or, what is the same thing, $b_1$ and $b_2$ each cross $p_2$ an even number of times, contrary to (18). Hence the broken line specified in the theorem exists.

20. THEOREM. *If two points $A$ and $B$ are connected by any broken line, finite or having at most the limit-vertices $A$ and $B$, then there is a subset of this broken line which forms a simple broken line connecting $A$ and $B$.*

PROOF. Inclose $A$ and $B$ in the small triangles $t_1$ and $t_2$ respectively. Let $A_1$ be a point of the broken line not within either triangle. From $A_1$ trace the broken line towards $B$ until we meet a point in the line already traced. Then a complete polygon has been traced, which we now omit from the broken line we are seeking. Since there are only a finite number of such polygons on $A_1 B_1$ exterior to the triangle $t_2$, we finally obtain a simple broken line connecting $A_1$ with a point within $t_2$. Since this is true for any triangle of which $B$ is an interior point, we have a simple broken line connecting $A_1$ and $B$. In the same manner we obtain a simple broken line connecting $A_1$ and $A$, and these together form the broken line required.

DEFINITIONS. *A set of points is bounded if it lies within a polygon.*

*A polygon is convex if for any line on which lies one of its sides there are no points of the polygon on one side of the line.*

21. THEOREM. *For any polygon $p$ there exists a convex polygon $p_1$ such that every interior point of $p_1$ lies within $p$.*

PROOF. About each limit-vertex $L_i$ of $p$ set a triangle $t_i$. Then there is a finite set of exposed segments. Connect every pair of end-points of these segments, forming a finite set of segments $[\sigma]$. Let $P$ be any interior point of $p$. Then there are points of $p$ and hence of $[\sigma]$ on both sides of every line through $P$. Draw any half-line from $P$ not meeting an end-point of $[\sigma]$. Then on this half-line there is a finite set of points of $[\sigma]$, and hence a last such point which lies on a segment $Q_1Q_2$. Then on one side of the line $Q_1Q_2$ there is no end-point and hence no point of $[\sigma]$, for if there were we should have a line meeting only one side of a triangle. Denote the angle $Q_1PQ_2$ by $\alpha_1$. From $P$ draw rays through the various end-points of $[\sigma]$ and order the angles thus formed, making a set $[\alpha_i]$. Since there are points of $[\sigma]$ on both sides of the line $PQ_2$, there are such points on that side of this line which is opposite the ray $PQ_1$. Hence there is an angle of $\alpha_i$, as $\alpha_2$, of which $PQ_2$ is a side, whose other side is on the

opposite side of $PQ_2$ from $PQ_1$, and within which there is no end-point of $[\sigma]$. Within $\angle \alpha_2$ construct a ray from $P$ meeting $[\sigma]$ in a last segment $Q_2Q_3$. Again on one side of the line $Q_2Q_3$ there is no point of $[\sigma]$. In this manner we continue until we reach $Q_1$. Then the polygon $Q_1Q_2$, $Q_2Q_3$, $\ldots$, $Q_nQ_1$ has the required properties.

## § 3. *Concerning a Sequence of Sets of Regions which Close down Uniformly on a Closed Set of Points.*

We now consider a plane in which Axioms I–VIII, $C$ of § 1 hold.

DEFINITIONS. *A set of points is "closed" if it contains all its limit-points.*

*A set of points is a "connected set" if at least one of any two complementary subsets contains a limit-point of points in the other set.*

*A "region" consists of an entirely open connected set together with any or all of those of its limit-points which are not points of the set.**

It is only in the presence of Axiom $C$ that a "closed" set as defined in the present paragraph differs from one not closed. The definition of "connectedness" given on page 293 may apply to a plane of Axioms I–VIII or to one of Axioms I–VIII and $C$, while the definition given in this section applies only in case Axiom $C$ is included. However, the latter definition of connectedness applies in cases where the former does not.

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* The terms "connected" and "region" have been defined variously. G. Cantor (*Mathematische Annalen*, Vol. XXI, p. 575) defines "connected" as follows, in terms of geometric congruence. A set of points $T$ is "zusammenhängend, wenn für je zwei Punkte $t$ und $t'$ derselben, bei vorgegebener beliebig kleiner Zahl $\varepsilon$ immer eine endliche Zahl Punkte $t_1$, $t_2$, $\ldots$, $t_\nu$ von $T$ auf mehrfache Art vorhanden sind, so dass die Entfernungen $\overline{tt_1}$, $\overline{t_1t_2}$, $\overline{t_2t_3}$, $\ldots$, $\overline{t_\nu t'}$ sämtlich kleiner sind, als $\varepsilon$."

W. H. Young, in his "The Theory of Sets of Points," p. 204, gives an equivalent definition in non-metrical terms: "A set of points such that, describing a region in any manner round each point and each limiting point of the set as internal points, these regions always generate a single region, is said to be a *connected set* provided it contains more than one point."

It will be noticed that these definitions make many sets connected which it would seem are not naturally so regarded. Thus, according to them the interior and exterior points of a circle or a triangle belong to the same connected set. A segment is connected though any set of isolated points is removed. In general, if from the ordinary continuum in space of any dimensions any set whatever which is nowhere dense is removed, the residue would form a connected set.

Schoenflies (*Mathematische Annalen*, Vol. LVIII, p. 209), following Jordan ("Cours d'Analyse," Vol. II, p. 25), first defines the notion of connectedness for a perfect set, "Eine perfekte Menge $T$ heisst zusammenhängen, wenn sie nicht in Teilmengen zerlegbar ist, deren jede perfekt ist." Also (p. 210), "*Die Komplimentärmenge M einer zusammenhängenden perfekten Menge T heisst zusammenhängend, resp. zusammenhängendes Gebiet, falls je zwei ihrer Punkte durch einen einfachen Weg verbindbar sind, der ihr ganz angehört.*" Schoenflies then remarks, "Diese Definition ist mit der Cantor'schen inhaltlich übereinstimmend," which is obviously not so. The example given above of the interior and exterior of a circle or a triangle, which under the Cantor definition belong to the same connected set, shows this, since under the Schoenflies definition just given these will not

We remark, in connection with this definition of region, that it is supposed to carry with it an implicit reference to the number of dimensions of the space that is considered. Thus if only the points of a line are considered, a segment of the line is a region. In a plane the interior of any polygon is a region, but this set does not form a region if it is considered in a three-dimensional space.

DEFINITION. *An infinite sequence of segments* $\{\sigma_i\}$ *of a line* $l$ *is said to "close down upon a point* $P$ *as a limit-point" if for every segment* $\sigma'$ *of* $l$ *containing* $P$ *there is a value of* $i$, $i = k$, *such that every segment* $\sigma_{k+j}$ $(i = 0, \ldots, \infty)$ *is contained in* $\sigma'$. $P$ *is said to be a limit-point of the sequence* $\{\sigma_i\}$.

1. THEOREM. *If a sequence of segments* $\{\sigma_i\}$ *close down upon a point* $P$ *as a limit-point, then there is no other point* $P' \neq P$ *which lies on every segment of* $\{\sigma_i\}$.

PROOF. Consider a segment containing $P$ of which $P'$ is one end-point. Then there is an infinitude of segments of $\{\sigma_i\}$ which lie on this segment and hence do not contain $P'$.

2. THEOREM. *For every point* $P$ *of a line* $l$ *there is a sequence of segments* $\{\sigma_i\}$ *on the line* $l$ *of which* $P$ *is a limit-point.*

PROOF. In the figure $l''$ is a half-line proceeding from $R$ in $l$ $(R \neq P)$, $l$ not containing $l''$. Let $S$ be a point on the same side of $l$ as $l''$, such that $S$ and $P$ are on opposite sides of $l''$. Connect $S$ and $P$, meeting $l''$ in $S'$. $Q$ is any point on $l''$ in the order $RS'Q$. Connect $P$ and $Q$ by the line $l'$ and let $P_1$ be any point of $l$ in the order $RPP_1$. From $S'$ project $P_1$ into $P_1'$ on $l'$, and from $S$ project $P_1'$ into $P_2$ on $l$, and so on. Continuing in this manner, using $S$ and $S'$ as centers of projection, we obtain a sequence of points $P_1$, $P_2$, $P_3$, ....

belong to the same connected set. Veblen (*Transactions of the American Mathematical Society*, Vol. VI, p. 91) uses the word "coherence" and defines the same as the Jordan-Schoenflies "Zusammenhang."

The term "region" is usually defined in substance as in the text of this paper, but from a variety of points of view and with varying degrees of complexity of statement. Veblen (*loco citato*, p. 85), however, defines "region" as "a set of points, any two of which are points of at least one broken line composed entirely of points of the set." This definition of "region" makes any broken line a region while an arc of a circle is not. The definition given by Young is (*loco citato*, p. 180): "A part of the plane which can be tiled over by a transitive set of triangles is called a domain or completely open region."... "The most general form of region consists of a domain together with some or all of its non-included limiting points."

The term "transitive," when applied to a set of triangles, is previously defined as follows: "Given a set of triangles, whose equivalent primitive triangles are $d_1$, $d_2$, ..., it may be that we can find a proper component of this set, $d_{i_1}$, $d_{i_2}$, ..., such that no triangle of this component overlaps with any but triangles of this component. If so, the set is said to be intransitive, otherwise transitive." The equivalent primitive triangles are triangles having rational points as vertices and containing the same interior points.

*We now assume as an axiom that $P$ is a limit-point of this sequence.**

A similar sequence of points $Q_1$, $Q_2$, .... on the segment $PR$ of which $P$ is also a limit-point gives the sequence of segments $\{P_i Q_i\}$ of which $P$ is a limit-point.

3. THEOREM. *If in the figure used in proving* (2) *a point $K$ is added in the order $PKP_1$, $l''$, $S$, $Q$ and $P_1$ remaining fixed, then, in the sequences $\{P_i\}$ and $\{K_i\}$ approaching $P$ and $K$ respectively* $(P_1 = K_1)$, *$P_i$ lies between $P$ and $K_i$ for $i \geqq 2$.*

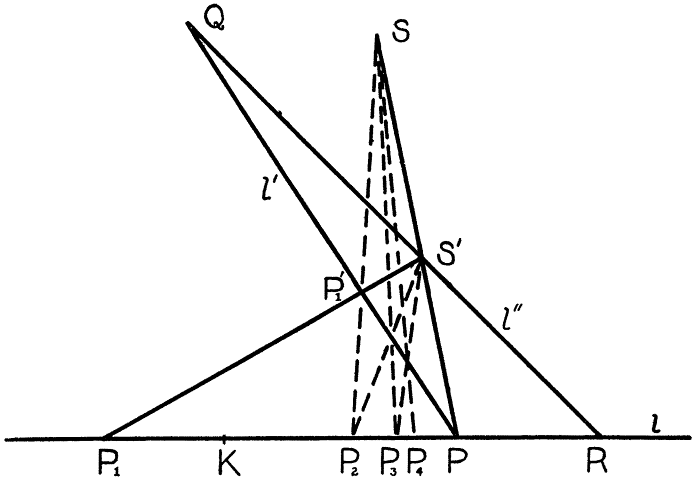


FIG. 4.

PROOF. This follows by mathematical inductions from elementary theorems on the interior and exterior of a triangle.

DEFINITION. *A sequence of sets of regions $\{[R]_i\}$ is said to close down uniformly upon a set of points $[P]$ if* (a) *every point $[P]$ is an interior point of some region of every set $[R]_i$ of $\{[R]_i\}$.*

(b) *For every finite set of regions $[R]'$ which contains every point of $[P]$ as interior points there is a value of $i$, $i = k$, such that every region of every set $[R]_{k+j}$ $(j = 0, \ldots, \infty)$ lies entirely within some region of $[R]'$.*

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* Von Staudt ("Geometrie der Lage," p. 50) uses essentially this construction in proving the fundamental theorem of projective geometry, but makes use of no axiom such as in the text. Klein (*Mathematische Annalen*, Vol. VI, p. 140) pointed out that the argument of Von Staudt is not conclusive. Klein uses a stronger axiom than the one here used; viz., that a limit-point (finite) of any sequence (bounded) exists. The axiom in its weaker form here used corresponds for projective geometry to the Archimedean axiom for metric geometry; viz., that for any two fixed segments $A_1 A_2$ and $\sigma$ one can apply $\sigma$ to $A_1 A_2$ a finite number of times and thus completely cover it. The theorem may of course be proved without the use of this special axiom if we assume the full axiom of continuity, p. 291.

4. THEOREM. *For every closed bounded set of points* $[P]$ *there is a sequence of finite sets of regions* $\{[R]_i\}$ *which closes down uniformly upon the set* $[P]$.

PROOF. (a) We consider first the case when the set is contained in a segment $AB$ of a line $l$. Select the points $P_1$ and $R$ in the order $P_1ABR$. Construct as in the proof of (2) a sequence of segments $\{A_i'B_i'\}_P$ for every point $P$ of the segment $AB$, using the same points $Q$, $S$, $R$ and $P_1$ for all points $P$ of $[P]$. Then each set $[A_i'B_i']$ ($i$ fixed) consists of an infinite set of segments of which every point of $[P]$ is an interior point. Since $[P]$ is a closed set, it follows by the Heine-Borel theorem* that there is a finite subset of $[A_i'B_i']$, $[AB]_i$, within which lie all points of $[P]$. We now show that $\{[AB]_i\}$ ($i = 1, \ldots, \infty$) is the required sequence of sets of regions. Let $[\sigma_i]$ be any finite set of segments such that every point of $[P]$ lies within at least one segment of the set. Consider any such segment $\sigma_i$ whose end-points are $C_iD_i$. Then, since $[P]$ is a closed set, either $C_i$ lies within a segment of $[\sigma_i]$ or there is a point $C_i'$ of $C_iD_i$ such that $C_iC_i'$ contains no point of $[P]$. In case $C_i$ lies on a segment of $[\sigma_i]$, a point $C_i'$ is chosen on $C_iD_i$ so that $C_iC_i'$ lies entirely within this segment. Points $D_i'$ are chosen in a similar manner with respect to $D_i$. The segment $C_i'D_i'$ having been thus constructed for a particular value of $i$, care is taken in constructing these segments for the other values of $i$ so that every point of $[P]$ shall lie within $[C_i'D_i']$. It is understood that all points $C_i$, $D_i$, $C_i'$ and $D_i'$ are in the order $P_1C_iD_iR$, and $P_1C_i'D_i'R$.

Consider now any particular segment of $[\sigma_i]$, as $C_1D_1$. In the sequence of points $\{P_n\}_{C_1'}$ approaching $C_1'$ there is by the definition of limit-point a value of $n$, $n_1$ such that all points $\{P_{n_1+j}\}_{C_1'}$ ($j = 0, \ldots, \infty$) lie on $C_1C_1'$. But by (3) $\{P_{n_1+j}\}_C$, where $C$ is any point of $C_1'C_2'$, lie on $C_1D_1$. We thus obtain such value of $n$, $n_i$ for every segment of $C_iD_i$. Let $N'$ be the largest of the finite set of numbers $n_i$. Then all points $[\{P_{N'+j}\}]$ ($j = 0, \ldots, \infty$) lie on a segment $C_iD_i$. Similarly we obtain the points $[\{P_{N''+j}\}]$ approaching the points of $[P]$ from the side on which $R$ lies. If $N$ is the greater of $N'$ and $N''$, then in the sequence $\{[AB]_i\}$, described above, every set $[AB]_i$ for $i \gtreqless N$ lies within $[\sigma_i]$.

(b) Next let $[P]$ be any closed bounded plane set. By (21), § 2, there is a

* For a proof of the Heine-Borel theorem in the plane, see paper by N. J. Lennes, *Bulletin of the American Mathematical Society*, Vol. XII (1906), pp. 395–398. The use of this theorem implies the use of the full continuity axiom. See O. Veblen, *Bulletin of the American Mathematical Society*, Vol. X (1904), pp. 436–439. The theorem under discussion is capable of proof without the use of this strong continuity, if it is not specified that each set of regions in the sequence $\{[R]_i\}$ should be finite.

convex polygon $p$ within which lie all points of $[P]$. It follows at once from (a) that there is a sequence of finite sets of segments which closes down uniformly on the set of all points of the polygon. Denote this sequence by $\{[AB]_i\}$. (We here include in "segment" the simple case of a broken line consisting of two segments whose common end-point is a vertex of the polygon.) Let $P_1$ and $P_2$ be two vertices of the polygon and the segment $P_1 P_2$ one of its sides. Connect each of the points $P_1$ and $P_2$ with the extremities of each segment of $[AB]_i$ for all values of $i$. Thus for each value of $i$ we obtain the set of polygonal regions into which these segments separate the region enclosed by the polygon $p$. Denote by $[R]_i$ a finite subset of this set of regions such that there are points of $[P]$ within every region of $[R]_i$. We now prove that $\{[R]_i\}$ is a sequence of sets of regions of the type specified in the theorem.

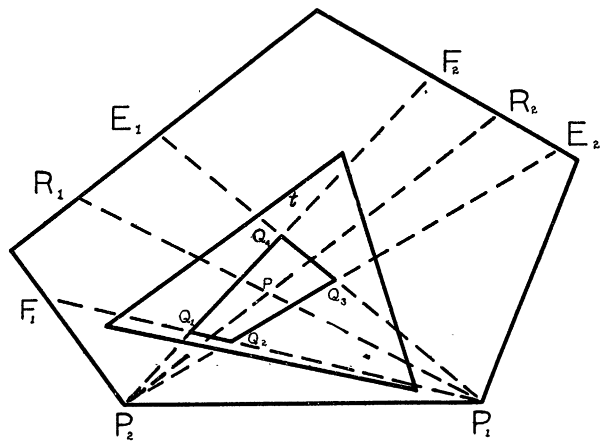


FIG. 5.

Let $[R]$ be any finite set of regions of which all points of $[P]$ are interior points. About every point $P$ of $[P]$ construct a triangle $t$ lying within a region of $[R]$. Through $P_1$, $P_2$ and $P$ construct segments $P_1 R_1$ and $P_2 R_2$ as shown in the figure. By means of these segments we can now construct $P_1 E_1$, $P_1 F_1$, $P_2 E_2$, $P_2 F_2$, such that the quadrilateral $Q_1 Q_2 Q_3 Q_4$ formed by them shall contain $P$ as an interior point and lie within the triangle $t$. Then the regions of the set consisting of all regions of the type $Q_1 Q_2 Q_3 Q_4$ lie within regions of $[R]$ and contain all points of $[P]$ as interior points. By the Heine-Borel theorem there is a finite subset $[R]'$ of this set of regions which fulfils these conditions. Consider now the set of segments $[EF]$ consisting of all segments of the polygon $p$ except the segments $P_1 P_2$ into which the points $E_1$, $F_1$, $E_2$, $F_2$, etc., separate it.

Then there is a value of $i$, $i=i_1$, such that, in the sequence of sets of segments $\{[AB]_i\}$, every segment of every set of $\{[AB]_i\}$ lies within a segment of $[EF]$ for every value of $i$ such that $i \geqq i_1$. Hence in the sequence of regions $\{[R]_i\}$ every region of every set for $i \geqq i_1$ lies within a region of the set $[R]'$, and hence within a region of $[R]$. Hence $\{[R]_i\}$ has the required properties.

## § 4. *Definition of Continuous Simple Curve.*

DEFINITION. *If every point of each of the sets $[P]'$ and $[P]''$ is a point of a set $[P]$, then we say that $[P]$ is the sum of the two sets $[P]'$ and $[P]''$ and write $[P]'+[P]''=[P]$. This does not imply that the sets $[P]'$ and $[P]''$ have no elements in common.*

1. THEOREM. *If each of the sets of points $[P]'$ and $[P]''$ is a connected set and has at least one point in common with the other, then the set $[P]=[P]'+[P]''$ is a connected set.*

PROOF. Let $[\overline{P}]$ and $[\overline{\overline{P}}]$ be any pair of complementary subsets of $[P]$. Then one of the following statements must hold:

(a) $[\overline{P}] \equiv [P]'$ or $[\overline{P}] \equiv [P]''$ and $[\overline{\overline{P}}] \equiv [P]''$ or $[\overline{\overline{P}}] \equiv [P]'$.

(b) There are points of at least one of the sets $[P]'$ and $[P]''$ in both $[\overline{P}]$ and $[\overline{\overline{P}}]$.

In case (a) $[\overline{P}]$ and $[\overline{\overline{P}}]$ have at least one point in common, whence either set contains a limit-point of points of the other set.

In case (b) it follows from the connected character of $[P]'$ and $[P]''$ that one of the sets $[\overline{P}]$ and $[\overline{\overline{P}}]$ contains a limit-point of points in the other set, whence the theorem is proved.

DEFINITION. *A continuous simple arc connecting two points $A$ and $B$, $A \neq B$, is a bounded, closed, connected set of points $[A]$ containing $A$ and $B$ such that no connected proper subset of $[A]$ contains $A$ and $B$.*

We speak of this arc as the arc $AB$ or $BA$, $A$ and $B$ being called the end-points of the arc. We note that a line-interval is an arc according to this definition.

2. THEOREM. *Every point $A_0$ of an arc $AB$, distinct from both $A$ and $B$, separates in a unique way the remaining points of the arc into two sets, one containing $A$ and the other containing $B$, such that the set containing $A$, together with $A_0$, forms an arc $AA_0$ and the set containing $B$, together with $A_0$, forms an arc $BA_0$. The arcs $AA_0$ and $BA_0$ have no point in common except $A_0$.*

PROOF. (a) By the definition of arc the points of the arc $AB$ apart from $A_0$ form at least one pair of complementary subsets, one set containing $A$ and the other containing $B$, such that neither set contains a limit-point of the other. Consider one* pair of such sets. Adjoin $A_0$ to each set and denote the set containing $A$ by $AA_0$ and the set containing $B$ by $BA_0$. We also denote the set forming the arc $AB$ by $[A]$.

(b) *The sets $AA_0$ and $BA_0$ are closed.*

By hypothesis all limit-points of $AA_0$ are points of $[A]$, $[A]$ being closed; and since $BA_0$ contains no limit-point of points of $AA_0$ (except possibly $A_0$), it follows that all such limit-points must be points of $AA_0$; that is, $AA_0$ forms a closed set. Similarly $BA_0$ is a closed set.

(c) *Each of the sets $AA_0$ and $BA_0$ is connected.*

Suppose that one of these sets, as $AA_0$, is not connected; *i. e.*, contains two non-vacuous complementary subsets neither one of which contains a limit-point of points of the other. To that one of these sets which contains $A_0$ add the set $BA_0$. Then we should have a pair of non-complementary subsets of $[A]$ neither of which contains a limit-point of the other, so that $[A]$ would not be a connected set.

(d) *The set $\begin{bmatrix} AA_0 \\ BA_0 \end{bmatrix}$ does not contain a connected proper subset containing $\begin{bmatrix} A \\ B \end{bmatrix}$ and $A_0$.*

If $AA_0$ contains a proper connected subset $\overline{AA_0}$ containing $A$ and $A_0$, then, by (1), $\overline{AA_0} + BA_0$ form a connected set containing $A$ and $B$, which is contrary to the definition of arc.

It follows from (a)–(d) that $AA_0$ and $BA_0$ are arcs. We refer to them as complementary arcs $a$ and $b$ of $AB$.

(e) *The set $[A]$ contains only one pair of complementary arcs connecting $A$ and $A_0$ and $B$ and $A_0$.*

Suppose there are two such pairs of arcs, $a$, $b$ and $a'$, $b'$. Since $a$ and $b$ contain together all points of $[A]$, it follows that $a$ and $b$ contain all points of $a'$. If not all points of $a$ are in $a'$, then the subset of $a'$ which is in $a$ is not connected. But adding any set of points from $b$ to this subset of $a$ must fail to make it connected, since neither of the sets $a$ and $b$ contains a limit-point of the other except $A_0$. Hence $a'$ would fail to be connected. In the same manner we show

* Under (e) we show that there is only one such pair.

that all points of $a'$ are points of $a$, whence $a$ and $a'$ are identical. In the same manner $b$ and $b'$ are identical.

DEFINITION. *A point "on an arc" is any point of the arc, including the end-points. A point "within an arc" is any point of the arc not an end-point.*

3. THEOREM. *If $A_0$ is a point within the arc $AB$, and $A_1$ any point within the arc $AA_0$, then the arc $A_1B$ contains every point of $A_0B$.*

PROOF. The arc $AA_1$ contains no point of $A_0B$, since $AA_1 + A_1A_0 = AA_0$ contains no point of $A_0B$ except $A_0$. Hence $A_1B$ contains all points of $A_0B$.

4. THEOREM. *Any two points of an arc determine uniquely an arc connecting them.*

DEFINITION. *Any point $A_1$ within an arc $AB$ is said to lie between the points $A$ and $B$ on the arc. We also say that the points $A$, $A_1$, $B$ are in the order $AA_1B$ or $BA_1A$ on the arc $AB$.*

5. THEOREM. *For any four points on an arc a notation may be so arranged that we shall have the orders $ABC$, $ABD$, $ACD$, $BCD$.*

PROOF. This is an immediate consequence of (4) and (3).

6. THEOREM. *If $A$ is an interior point of a polygon and $B$ an exterior point, then every arc $AB$ contains a point of the polygon.*

PROOF. Suppose there are two complementary subsets of the arc $AB$ such that one lies outside the polygon and the other inside the polygon; then, by (16), § 2, neither of these sets contains a limit-point of the other and hence the arc $AB$ would not be connected.

7. THEOREM (Ordinal Continuity of an Arc). (a) *If $A_1$ and $A_2$ are any two points of an arc, then there is a point $A_3$ on the arc in the order $A_1A_3A_2$.*

(b) *If $[A]'$ and $[A]''$ are complementary subsets of the points of an arc $AB$ such that no point in either set lies between points of the other set on the arc, then aside from $A$ and $B$ there is one and only one point of the arc which does not lie between points of either set.*

PROOF. (a) is a direct consequence of (4) and the *connected* property of an arc.

(b) Let $[A]'$ and $[A]''$ be any pair of complementary subsets of an arc $AB$ such that no pair of either lies between points of the other. Then there is a point $A_0$ in one of these sets, as $[A]'$, which is a limit-point of points in the other set. The points of $[A]''$ lie entirely on one of the arcs $AA_0$ and $BA_0$, as $BA_0$, for otherwise we should have the point $A_0$ of $[A]'$ between points of $[A]''$. Suppose now there is a point $A_1$ of $[A]'$ on the arc $BA_0$; then, since $A_0$ is a limit-point of $[A]''$, there are points of $[A]''$ between $A_1$ and $A_0$, which are both of $[A]'$. Hence there is no point of $[A]'$ on $BA_0$, and $A_0$ is therefore the required point.

8. THEOREM (Geometric Continuity of an Arc). *If $A_0$ is any point of an arc $AB$, and $t_1$ any triangle containing $A_0$ as an interior point, then (in case $A_0 \neq A$) there is a point $A_1$ on the arc $AA_0$ and (in case $A_0 \neq B$) a similar point $B_1$ on the arc $BA_0$ such that every point of the arc $A_1 B_1$ lies within $t_1$.*

PROOF. Let $t_2$ be any triangle within $t_1$ also containing $A_0$ as an interior point. We consider the arc $AA_0$. Suppose $A$ exterior to $t_2$. Then by (6) there are points of $AA_0$ on $t_2$. If the theorem does not hold, then, for any point $A_2^{(1)}$ of $AA_0$ on $t_2$, there are by (6) points of $A_2^{(1)}A_0$ on $t_1$. Let $A_1^{(1)}$ be such a point. Further, there is a point $A_2^{(2)}$ of the arc $A_1^{(1)}A_0$ on $t_2$, a point $A_1^{(2)}$ of the arc $A_2^{(2)}A_0$ on $t_1$, etc. In this manner we obtain an infinite sequence of points $\{A_1^{(i)}\}$ of $AA_0$ on $t_1$ and a sequence $\{A_2^{(i)}\}$ of $AA_0$ on $t_2$. Let $A_1'$ be a limit-point of $\{A_1^{(i)}\}$ and $A_2'$ a limit-point of $\{A_2^{(i)}\}$.*

The points $A_1'$ and $A_2'$ can not lie on an arc $A_1^{(i)}A_2^{(i)}$, for in that case such arc would contain a limit-point of the arc $A_1^{(i+1)}A_0$, which is impossible. Further, $A_1'$ and $A_2'$ are points of the arc $AA_0$. Suppose we have the order $A_1'A_2'A_0$. Then all the arcs $A_2^{(i)}A_2^{(i+1)}$ lie on the arc $AA_1'$; for suppose one such arc, as $A_2^kA_2^{(k+1)}$, lies on $A_1'A_0$, then all subsequent arcs of the sequence lie on this arc, and hence $A_1'$ can not be a limit-point of points of these arcs. But if all arcs of the sequence $A_2^{(i)}A_2^{(i+1)}$ lie on $AA_1'$, then $A_2'$ can not be a limit-point of these arcs. Similarly for the order $A_2'A_1'A_0$.

9. THEOREM. *If $A_0$ is any point of an arc $AB$, and $t_1$ any triangle containing $A_0$ as an interior point, then there exists a triangle $t_2$ containing $A_0$ as an interior point such that every arc of $AB$ which connects $A_0$ with a point of $t_2$ lies entirely within $t_1$.*

PROOF. Let $A_1$ be a point on the arc $AA_0$ such that no point of the arc $A_1A_0$ is on or exterior to the triangle $t_1$, (8), and $B_1$ a similar point on $A_0B$. Then $A_0$ is not a limit-point of points on the arcs $AA_1$ and $BB_1$. Hence, by definition of limit-point there is a triangle $t_2$ within $t_1$, and containing $A_0$ as an interior point, such that there is no point of $AA_1$ and $BB_1$ on or within $t_2$. Hence $t_2$ is the required triangle.

10. THEOREM. *The points of any two arcs may be set into complete one-to-one correspondence preserving order.*†

* The existence of the points $A'_1$ and $A'_2$ follows from axiom $C$ by well-known argumentation.

† Professor Veblen has proved ("Theory of Plane Curves in Non-Metrical Analysis Situs," *Transactions of the American Mathematical Society*, Vol. VI (1905), pp. 83–98) that two sets of points possessing the order relations specified under (7) and (8) may be thus set into a one-to-one correspondence. Veblen's proof consists in showing that any set having these properties contains a numerably infinite set of points which is everywhere dense in the set, and then applying a theorem of G. Cantor (*Mathematische Annalen*, Vol. XLVI (1895), pp. 481–512) to the effect that all sets having this property together with those given by the theorems (2), (7), (8) may be thus set into correspondence. However, Veblen's proof involves metric relations, inasmuch as he makes use of equal segments.

PROOF. Let $\{[t]_i\}$ be an infinite sequence of finite sets of triangular regions closing down uniformly on the points of an arc $AB$ (see § 3). Let $[A]_i$ be a finite set of points of $AB$ containing at least one point within each triangle of the set $[t]_i$ such that $[A]_i$ contains all points of $[A]_{i-1}$ for all values of $i$. Then the infinite sequence $\{[A]_i\}$ contains a numerably infinite set of points which is everywhere dense on the arc $AB$. The theorem now follows from the theorem of G. Cantor cited in the foot-note. The proof may also be completed very easily as follows. For any two arcs $AB$ and $A'B'$ the sets $\{[A]_i\}$ and $\{[A']_i\}$ may obviously be set into correspondence preserving order. In order that $[A]_i$ and $[A']_i$ shall contain the same number of points we add the requisite number of points to one of them. Let $A_0$ be any point of $AB$ not of $\{[A]_i\}$. Then $A_0$ separates $\{[A]_i\}$ into two sets neither one of which contains a point between points of the other. There will then be a corresponding division of $\{[A']_i\}$, whence, by (7), there is a point $A'_0$ which we now set in correspondence with $A_0$. In this manner all points of the two arcs are set into a one-to-one correspondence. That order is preserved is obvious.

## § 5. *The Frontier of a Region.*

DEFINITIONS. *Consider an entirely open bounded region $R$ enclosed in a polygon $p$ such that there is no limit-point of $R$ on $p$. Denote by $[E']$ all points of the plane accessible from $p$ with respect to $R$, and by $[F]$ all common limit-points of $[E']$ and $R$. Denote by $[I]$ all points of the plane which are contained in neither of the sets $[E']$ and $[F]$, and by $[E]$ all points of $[E']$ not of $[F]$.*

*$[F]$ is called the "frontier" of the set $[I]$. $[I]$ is the interior set of $[F]$, and $[E]$ its exterior set.*

*A point $F_1$ of $[F]$ is said to possess exterior accessibility if there exists a finite or continuous infinite broken line connecting it with a point of $[E]$, and to possess internal accessibility if there exists a finite or continuous infinite broken line connecting it with a point of $I$, the broken line in either case containing no point of $[F]$ except $F_1$.*

1. THEOREM. *If every point of a frontier $[F]$ possesses external accessibility, it separates the remaining points of the plane into two connected sets $[E]$ and $[I]$.*

PROOF. (a) By definition any broken line connecting a point of $[E]$ with a point of $[I]$ meets $[F]$, for otherwise some point of $[I]$ would be accessible from a point of the bounding polygon $p$.

(b) Any two exterior points are mutually accessible, since all such points are accessible from points of $p$.

(c) Any two interior points $I_1$ and $I_2$ are mutually accessible with respect to $[F]$ if both lie in $R$. This is an immediate consequence of the entirely open, connected and bounded character of $R$.

(d) Any two interior points $I_1$ and $I_2$ are mutually accessible with respect to $[F]$. If one of these points, as $I_1$, is not a point of $R$, we need only to prove that $I_1$ is accessible from some point of $R$ with respect to $[F]$. Join $I_1$ to $F_1$ and $F_2$ of $[F]$ by means of segments $I_1F_1$ and $I_1F_2$, neither of which contains a point of $[F]$. That such segments exist is an immediate consequence of axiom $C$ and the closed character of $[F]$. Connect $F_1$ and $F_2$ with points of $p$ by means of continuous simple broken lines, (20), § 2, containing no points of $[F]$ except $F_1$ and $F_2$. By (17), § 2, these broken lines, together with the polygon $p$, form two polygons having no interior points in common. There are points of $[F]$ and hence, by (16), § 2, points of $R$ within each polygon, for otherwise $I_1$ would be accessible from $p$. Since $R$ is a connected set, there must be points of $R$ on each polygon. But all segments of these polygons except $I_1F_1$ and $I_1F_2$ lie in $[E]$. Hence there are points of $R$ on one of the segments $I_1F_1$ and $I_1F_2$ whence $I_1$ is accessible from some point of $R$.*

2. THEOREM. *If every point of a frontier $[F]$ possesses both interior and exterior accessibility, then any two points $F_1$ and $F_2$ of $[F]$ may be connected by two simple broken lines, one in $[I]$ and one in $[E]$, and these two broken lines form a polygon which separates the remaining points of $[F]$ into two sets each of which is a continuous arc connecting $F_1$* and $F_2$.

PROOF. The existence of such broken lines is an immediate consequence of the twofold accessibility of every point of $[F]$ and the connected character of $[E]$ and $[I]$ and (20), § 2. By definition these broken lines form a polygon $p'$. Let $E_1$ be an exterior point of $[F]$ on $p$ and $I_1$ an interior point of $[F]$ on $p'$. Then there are points of $[F]$ both exterior and interior to $p'$, for otherwise $I_1$ and $E_1$ would be mutually accessible with respect to $[F]$, (12), § 2. Denote by $[F]'$ the points of $[F]$ within $p'$, together with the points $F_1$ and $F_2$.

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* We note that (a), (b), (c) follow from the definition of frontier without the use of the special assumption of exterior accessibility. That (d) does not follow without this special assumption is shown by the following example: Consider a circle with two spirals, each going around the circle an infinite number of times and approaching it but having no point in common. Connect these spirals by means of a segment. Then the spirals, together with the segment, enclose a region $R$ which contains no interior point of the circle. The set $[I]$ defined by this region contains also the interior of the circle and is thus not connected.

(a) Since $[F]$ is a closed set, it follows from (16), § 2, that $[F]'$ is closed.

(b) There is no connected proper subset of $[F]'$ containing $F_1$ and $F_2$; for if there is such subset, let $F'$ be a point of $[F]'$ but not of the connected subset. Then $F'$ may be connected with $E_1$ and $I_1$ by means of a broken line connecting no other point of $[F]$. It is evident that these broken lines may be so chosen as to lie entirely within $p'$. Then two polygons are formed such that the points of $[F]'$, except $F_1$, $F_2$ and $F'$, lie within one or the other of the polygons. Since $F_1 \neq F_2$ it follows that there are points of $[F]'$ within each polygon, and by (16), §2, these do not form one connected set unless $F'$ is included.

(c) $[F]'$ is a connected set. Suppose $[F]'$ is not connected and that $[F]_1'$ is any closed subset of it which contains no limit-point of the complementary set $[F]_2'$. Suppose $A$ is a point of $[F]_1'$. About $A$ set a triangle containing no point of $[F]_2'$. About every other point of $[F]_1'$ set a triangle lying within $p'$ and on or within which lie no points of $[F]_2'$. Then, by the Heine-Borel Theorem, there is a finite subset of these triangles within which lie all points of $[F]_1'$. By (15), § 2, there is a finite polygon $p''$ which incloses this set of points, but which does not contain the point $F_2$. Hence, by (19), § 2, there is a broken line of $p''$ lying within $p'$, connecting a point on the exterior broken line of $p$ with a point of the interior broken line of $p'$ and not meeting $[F]$. But this contradicts (1).*

DEFINITION. *The set of points consisting of two continuous arcs, each connecting a pair of distinct points $A$ and $B$ and having no other point in common, is called a simple closed Jordan curve, or simply a Jordan curve. We denote it by $j$.*

3. THEOREM. *If every point of a frontier $[F]$ possesses both internal and external accessibility, it is a Jordan curve.*

PROOF. This is an immediate consequence of the definition of Jordan curve and (2).

## § 6. *Separation of the Plane by a Jordan Curve.*

In § 4 we showed that the points of a continuous arc, as there defined, may be set into a one-to-one correspondence with the points of a straight line interval preserving order. In § 5 a proof is given that the frontier of a region accessible

* It does not follow that the points of a frontier possess internal accessibility even if they possess external accessibility. This is shown by the following well-known example: The point $\left(\frac{2}{\pi}, 1\right)$ on the curve $y = \sin \frac{1}{x}$ is connected to the point (0, 1) by means of a broken line containing no point of the curve.

from both exterior and interior points is a Jordan curve; that is, a curve consisting of two non-intersecting arcs connecting the same two points. In the present section a proof is given of the converse theorem; viz., that a Jordan curve separates the remaining points of the plane into two entirely open sets.*

For the purpose of studying a Jordan curve, denoted by $j$, we construct a polygon $p$ having the following properties: Two points $P_1$ and $P_2$ of $j$ are on $p$, and all the remaining points of $j$ are interior points of $p$. To construct such a polygon let $p'$ be any convex polygon, (21), § 2, within which lie all points of $j$. Since $j$ is a closed set of points, we obtain by the axiom of continuity an angle with its vertex $A_1$ one of the vertices of $p'$, such that there are points of $j$ on each side of the angle but no points of $j$ exterior to the angle. Let $P_1$ and $P_2$ be points of $j$, one on each side of the angle. Connect $P_1$ with the polygon $p'$ by means of two segments exterior to the angle, and similarly for $P_2$. Then these four segments, together with a properly chosen subset of $p'$, form the required polygon $p$.

The points $P_1$ and $P_2$ separate the polygon $p$ into two broken lines which we denote by $b_1$ and $b_2$, and the curve $j$ into two arcs which we denote by $a_1$ and $a_2$. The following propositions are stated in terms of this notation.

1. THEOREM. *If an arc $a_1$ with end-points $P_1$ and $P_2$ on a polygon $p$ lies entirely within $p$, except $P_1$ and $P_2$, then some but not all interior points of $p$ are accessible from $b_1$ with respect to $a_1$ by means of broken lines lying within*

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* This classic theorem was first stated and proved by Jordan (C. Jordan, "Cours d'Analyse," Paris, 1893, 2d ed., p. 92). For a brief characterization of the literature on this subject, see O. Veblen, *Transactions of the American Mathematical Society*, Vol. VI, pp. 83–98.

The definition given by Jordan in terms of analytic geometry is as follows:

Consider two equations

$$\begin{cases} x = f(t), \\ y = \varphi(t), \end{cases}$$

where $f(t)$ and $\varphi(t)$ have the following properties:

(a) $t$ takes all values of an interval $a \ldots . b$.

(b) $f(t)$ and $\phi(t)$ are continuous functions of $t$ on the interval $a \ldots . b$.

(c) $f(a) = f(b)$ and $\phi(a) = \phi(b)$.

(d) There is no pair of distinct values of $t$, $t_1$ and $t_2$ such that $f(t_1) = f(t_2)$ and $\phi(t_1) = \phi(t_2)$.

The curve defined by these equations we may now readily show is identical with the Jordan curve defined in § 5.

Let $P_1'$ and $P_2'$ be any two points of $a \ldots . b$, $P_1' \neq P_2'$, not both being end-points of the interval. These points separate the interval into two sets, one set consisting of the points lying between the two points, and the other set consisting of the remaining points of the interval. It is clear that the points $P_1$ and $P_2$ of the curve corresponding to $P_1'$ and $P_2'$ of the interval separate the curve into two parts and that each part is a continuous simple arc; viz., each is a closed, bounded, connected set containing $P_1$ and $P_2$ which has no connected proper subset connecting these points.

*$p$. No point of $b_2$ is thus accessible from $b_1$, and any point accessible from $b_1$ is not accessible from $b_2$.*

PROOF. (a) Since no points of $a_1$, except $P_1$ and $P_2$, are limit-points of $b_1$, it follows that there are interior points of $p$ accessible as stated in the theorem.

(b) If every interior point of $p$ is so accessible, it follows that points of $b_2$ are accessible, since the points of $b_2$ are not limit-points of $a_1$. But if a point of $b_2$ is thus accessible, we shall have two polygons with no interior point in common, (17), § 2, within each of which lie points of $a_1$. Since there are no points of $a_1$ except $P_1$ and $P_2$ on these polygons, it follows, (16), § 2, that $a_1$ is not a connected set of points, which is contrary to the definition of $a_1$. If the same point not on $a_1$ were accessible from both $b_1$ and $b_2$, then a point on $b_2$ would be accessible from $b_1$.

Denote the set of points within $p$ and not of $a_1$ which are accessible from $b_1$ by $[S]_1$, and the remaining points within $p$ and not of $a_1$ by $[S]_2$.

2. THEOREM. *If any point of the arc $a_2$ lies in one of the sets $[S]_1$ and $[S]_2$ into which $a_1$ separates the remaining interior points of $p$, then all points of $a_2$ lie within this set; and hence if a point of $a_1$ is accessible from $b_1$, no point of $a_2$ is accessible from $b_1$.*

PROOF. The theorem follows from the connected character of $a_2$ when we establish that neither of the sets $[S]_1$ and $[S]_2$ has a limit-point of the other. If $Q$, a point of $[S]_1$, is a limit-point of points in $[S]_2$, we can construct a triangle containing $Q$ as an interior point but no point of $a_1$, $a_1$ being a closed set and hence $Q$ not a limit-point of $a_1$. But there are points of $[S]_2$ within this triangle, and hence $Q$ is accessible from $b_2$, which is contrary to the definition of $[S]_1$ and $[S]_2$.

3. THEOREM. *There is a set of points, not on $j$, which is not accessible from $p$ by means of any broken line whatever which contains no point of $j$.*

PROOF. Connect a point $H$ on $b_1$ with a point $K$ on $b_2$ by means of a broken line lying within $p$. Let $a_1$ be the arc accessible from $b_1$. Since $a_1$ is closed, it follows that there is a point $P_1$, on $a_1$ and the broken line $HK$, such that there is no point of $a_1$ on the broken line $HK$ between $P_1$ and $K$. There are points of $a_2$ between $P_1$ and $K$, for otherwise the point $P_1$ on $a_1$ would be accessible from $b_2$. Since $a_2$ is closed, there is a point $P_2$ on $a_2$ and the broken line $HK$, such that there is no point of either $a_1$ or $a_2$ between $P_1$ and $P_2$. Since $P_1$ is not accessible from $b_2$ and $P_2$ is not accessible from $b_1$, it follows that the points of the broken line $P_1 P_2$ are not accessible from either $b_1$ or $b_2$. Hence these points are of the required type.

Definition. *Every point of the plane, not of $j$, which is accessible from points of $p$ by any broken line whatsoever, is an exterior point of $j$; a point not so accessible is an interior point.*

4. Theorem. *The exterior and interior character of a point with respect to a given curve, as here defined, is independent of the polygon $p$.*

Proof. Consider any two polygons $p'$ and $p''$ such that no point of the curve $j$ is exterior to either of them. Since any point on either polygon can be connected with any point on the other by a broken line containing no point of $j$, it follows that any point of the plane which can be connected with a point of one of these polygons can be connected with the other.

5. Theorem. *Every point of $j$ is accessible from $p$ with respect to $j$.*

Proof. Let $A$ be any point of $a_1$. Consider a sequence of triangles $t^i$ closing down upon $A$ as a limit-point, (4), § 3, and having the following properties:

(a) Every triangle lies within the polygon $p$.

(b) Every point of the arc $P_1 A$ which lies between two points of $t_i$ is an interior point of $t_{i-1}$ $(i = 2, \ldots, \infty)$, (9), § 4.

(c) $t_i$ lies within $t_{i-1}$ $(i = 2, \ldots, \infty)$.

Let $A_i$ be a set of points on the arc $P_1 A$ such that $A_i$ lies on the triangle $t_i$. About every point of $P_2 A$ except $A$ construct a triangle $t$, forming a set $[t]$ having the following properties:

(a) No point of the arc $AP_2$ is on or within one of the triangles.

(b) Those triangles of $[t]$ which are constructed about the points of the arc $A_{i-1} A_i$ are interior to $t_{i-2}$ and exterior to $t_{i+1}$.

The triangles containing as interior points the points of the arc $A_{i-1} A_i$ contain, according to the Heine-Borel Theorem, a finite subset of triangles such that every point of $A_{i-1} A_i$ is an interior point of the set. Since the arc $A_{i-1} A_i$ is a connected set, the set of triangles is overlapping, whence, by (15), § 2, there is a finite polygon within $t_{i-2}$ and exterior to $t_{i+1}$ within which lie all points of $A_{i-1} A_i$. That no point of the arc $A_{i+1} P_2$ lies within this polygon follows from the connected character of the arc and the two facts that the polygon contains no point of $A_{i+1} P_2$ and that $P_2$ is surely exterior to the polygon.

We thus obtain an infinite sequence $\{p_i\}$ of finite polygons having the following properties:

(a) $p_i$ contains all points of $P_{i-1} P_i$ as interior points $(i = 2, \ldots, \infty)$.

(b) $p_i$ lies within $t_{i-2}$ and exterior to $t_{i+2}$ $(i = 3, \ldots, \infty)$. ($p_1$ contains the arc $P_1 A_1$ and is exterior to $t_2$.)

Then by (15), § 2, a certain subset of $p_1$ and $p_2$ forms a polygon $p^{(1)}$ which contains all points of $P_1 A_2$ as interior points. Similarly a certain subset of $p^{(1)}$ and $p_3$ forms a polygon $p^{(2)}$ which contains all points of $P_1 A_3$ as interior points; and, in general, a certain subset of $p^{(i)}$ and $p_{i+2}$ forms a polygon $p^{(i+1)}$ which contains all points of $P_1 A_{i+2}$ as interior points, and within which lie no points of the arc $A_{i+3} P_2$. Also every segment of $p^{(i+1)}$ which is exterior to $t_i$ is a segment of $p^{(i)}$. Further, there is no point of $a_1$ on $p^{(i+1)}$ except within the triangle $t_i$. Tracing the polygon $p^{(i+1)}$ from a point on the polygon $p$, let $Q_i$ be the first point reached on $p_i$. Then we obtain a sequence $\{Q_i Q_{i+1}\}$ of finite broken lines forming an infinite broken line such that there are only a finite number of its segments exterior to any triangle of the sequence $\{t_i\}$, while there are segments of the sequence within every such triangle. Hence $A$ is accessible from points on both $b_1$ and $b_2$ by means of two distinct broken lines. We now observe that one of these broken lines lies entirely in $[S]_1$ and the other in $[S]_2$. If then $a_2$ lies in $[S]_2$, $A$ is accessible by means of the broken line lying in $[S]_1$.

6. THEOREM. *There is an interior point of $j$ from which all its points are accessible with respect to $j$.*

PROOF. By (3) there exists an interior point $I$ of $j$, and hence an interior segment with its end-points $Q_1$ and $Q_2$, one on $a_1$ and the other on $a_2$. Connect points on $b_1$ and $b_2$ with $Q_1$ and $Q_2$, respectively, by means of broken lines containing no points of $j$ except $Q_1$ and $Q_2$. Denote by $a_1'$ and $a_2'$ the arcs into which $Q_1$ and $Q_2$ separate $j$. Then, by (17), § 2, we have two polygons $p$ and $p$ one of which contains the arc $a_1'$ and the other the arc $a_2'$. Then, by (5), every point of $a_1'$ and also of $a_2'$ is accessible from $I$.

7. THEOREM. *A Jordan curve separates all the remaining points of the plane into two connected sets.*

PROOF. The uniqueness of the exterior and interior sets as defined on p. 317 is established in (4). That no two points, one interior and the other exterior, are mutually accessible with respect to $j$ is a direct consequence of the definition of these sets. That any two exterior points are mutually accessible follows from the fact that both are accessible from points on $p$.

Let $I_1$ and $I_2$ be any two interior points. Through $I_1$ pass a broken line meeting $j$ in only two points, as in the proof of (6), and producing them to reach $p$. Then we have two polygons $p_1'$ and $p_2'$ within each of which lie of points of $j$. Connect $I_2$ with a point of that polygon within which $I_2$ does not lie, say $p_1'$. This connecting broken line must meet $p_1'$ in an interior point whence $I_1$ on this broken line is accessible from $I_2$, which completes the proof of the theorem.

## § 7. *Concerning a Set of Simple Continuous Arcs Having a Simple Continuous Arc as a Limit.*

Denote by $R$ a closed bounded set of points in which two points $A$ and $B$ are connected by an infinite set $[a]$ of simple continuous arcs, $A$ and $B$ being the end-points of each arc of $[a]$.*

The arcs of the set $[a]$ are assumed to satisfy the following condition, which we call *uniform continuity of the arcs with respect to the set.*

*If $P$ is any point of $R$ and $t_1$ is any triangle containing $P$ as an interior point, there exists a triangle $t_2$ within $t_1$ containing $P$ as an interior point, such that no arc of $[a]$ contains a point of $t_1$ between points of $t_2$.*†

Let $[A]$ denote the set of points consisting of all points of the arcs of $[a]$, together with their limit-points. Let $\{[t]_i\}$ be a sequence of sets of triangles enclosing triangular regions which close down uniformly upon the set $[A]$. Denote generically by $m_i$ the sum of the number of triangles of the sets $[t]_1$, $[t]_2, \ldots, [t]_i$ of $\{[t]_i\}$. Then the number of different triangles of $[t]_i$ within which lie points of any arc of $[a]$ is less than $m_i$.

We now proceed to describe a set of points on a certain subset of the arcs $[a]$ which bear a special relation to the set of triangles $[t]_i$ for a definite fixed value of $i$. On each arc of $[a]$ select $m_i$ points forming a set $P_{[a]_i, j}$ having the following properties:

(a) On each arc at least one point lies within every triangle of $[t]_i$ within which that arc contains points.

(b) On each arc the order of the points is indicated by the subscript $j$; viz., the points are in the order

$$A = P_{[a]_i, 1}; \quad P_{[a]_i, 2}; \; \ldots; \quad P_{[a]_i, m_i - 1}; \quad P_{[a]_i, m} = B.$$

For a fixed value of $j$ there is then an infinitude of points of $P_{[a]_i, j}$, one on each arc of $[a]$ which has one or more limit-points. Consider this set for $j = 2$. Let $A_{i, 2}$ be a limit-point of the set $P_{[a]_i, 2}$, and let $P'_{[a]_i, 2}$ be a subset of $P_{[a]_i, 2}$ such

* Obviously there are closed and connected sets which contain no continuous arcs connecting certain pairs of their points. This discussion does not apply to such sets and such pairs of points. In case there is only one arc in $R$ connecting $A$ and $B$, or in case all such arcs partly coincide, then the arcs of $[a]$ coincide wholly or in part.

† This condition is the non-metrical equivalent of a condition stated by G. Ascoli in the usual metric terms [G. Ascoli: "Accademia die Lincei," (1884)]. The theorem of Ascoli, which seems to have been neglected, is capable of extension and important applications. This subject will be treated in a forthcoming paper by the writer. The present paper was written without my being aware of the work of Ascoli.

that $A_{i,2}$ is the only limit-point of the set. Let $[a]_{i,2}$ be the set of arcs of $[a]$ on which lie the points $P'_{[a]_i,2}$. For $j=3$ we now select an infinite subset $P'_{[a]_i,3}$ of $P_{[a]_i,3}$ which has only one limit-point $A_{i,3}$, and all of whose points lie on a subset $[a]_{i,3}$ of the set $[a]_{i,2}$.* We note that $A_{i,2}$ is a limit-point of that subset of $P_{[a]_i,2}$ which lies on arcs of $[a]_{i,3}$.

In general, for any fixed value of $j$ we select an infinite subset $P'_{[a]_i,j}$ of $P_{[a]_i,j}$ which has only one limit-point $A_{i,j}$, and which lies on arcs of $[a]_{i,j-1}$. That subset of $[a]_{i,j-1}$ on which lie points of $P'_{[a]_i,j}$ we denote by $[a]_{i,j}$.

Finally we denote by $[a]_i$ the set of arcs of $[a]$ which are arcs of all the sets $[a]_{i,j}$ $(j=2, \ldots, m_i-1)$. The points of $P'_{[a]_i,j}$ $(j=2, \ldots, m_i-1)$ which lie on arcs of $[a]_i$ we denote by $P_{[a]_i,j}$. Clearly $A_{i,j}$ is a limit-point of $P_{[a]_i,j}$ for all admitted values of $j$.

Then on each arc, as $a_{i,1}$, of $[a]_i$ we have a set of $m_i$ points $P_{a_{i,1},j}$ in the order

$$A = P_{a_{i,1},1}; \quad P_{a_{i,1},2}; \quad P_{a_{i,1},3}; \quad \ldots; \quad P_{a_{i,1},m_i-1}; \quad P_{a_{i,1},m_i} = B.$$

The $m_i$ sets of points $P_{[a]_i,j}$ ($i$ fixed; $j=1, \ldots, m_i$) have the limit-points $A_{i,j}$ and no others.

In a similar manner for the same fixed $i$ we now obtain a set of $m_{i+1}$ sets of points $P_{[a]_{i+1},j}$ $(j=1, \ldots, m_{i+1})$ and a set of arcs $[a]_{i+1}$ having the following properties:

(a) The set of arcs $[a]_{i+1}$ is a subset of $[a]_i$.

(b) $P_{[a]_{i+1},j}$ contains all those points of $P_{[a]_i,j}$ which lie on arcs of $[a]_{i+1}$.

(c) $P_{[a]_{i+1},j}$ consists of $m_{i+1}$ points on each arc of $[a]_{i+1}$, and contains on each arc a point within each triangle of $[t]_{i+1}$ within which are points of the arc.

(d) For any fixed $j$, $P_{[a]_{i+1},j}$ has only one limit-point $A_{i+1,j}$.

(e) The set of points $A_{i+1,j}$ contains all points of the set $A_{i,j}$.

The notion *order* may now be associated as follows with the set of points $A_{i,j}$ ($i=1, \ldots, \infty$; and for each value of $i$, $j=1, \ldots, m_i$). Let $A^{(1)}$ and $A^{(2)}$ be any two such points, each distinct from $A$ and from $B$. Then there is a value of $i$, as $i=h$, such that for certain values of $j$, as $j=k$ and $j=l$, $A^{(1)}$ is

* We note that a subset may contain all the elements of the original set; that is, the word subset does not necessarily mean proper subset. We also note that for fixed $j$ the points $P_{[a]_i,j}$ may all coincide. In that case $A_{i,j}$ coincides with these points.

the limit-point of the $P_{[a]_h, k}$ and $A^{(2)}$ is the limit-point of $P_{[a]_h, l}$. Then the points $A$, $A^{(1)}$, $A^{(2)}$, $B$ are said to be in the same order as $A$; $P_{[a]_h, k}$; $P_{[a]_h, l}$; $B$ on the various arcs of $[a]_h$.

DEFINITION. *We now consider a set of points $[A]$ consisting of all points $A_{i,j}$ ($i=1, \ldots, \infty$; and for each value of $i$, $j=1, \ldots, m_i$) together with their limit points.*

1. THEOREM. *The set $[A]$ is a simple continuous arc connecting the points $A$ and $B$.*

PROOF. (a) *The set $[A]$ is closed by definition.*

(b) *The set $[A]$ is connected.*

Suppose that $[A]$ contains two complementary subsets neither of which contains a limit-point of the other. Denote these sets together with their non-contained limit-points by $[A]_1$ and $[A]_2$ respectively. By the Heine-Borel Theorem we obtain two finite sets of triangles $[t]_1$ and $[t]_2$ such that every triangle of either set is entirely exterior to every triangle of the other set, and such that all points of $[A]_1$ lie within triangles of $[t]_1$ and all points of $[A]_2$ within triangles of $[t]_2$.

Since points of $A_{i,j}$ must lie within each triangle of $[t]_1$ and of $[t]_2$, and since for all values of $i$ equal to or greater than a certain fixed number $k$ only a finite number of arcs of $[a]_i$ can fail to contain points within a triangle which contains points of $A_{i,j}$, and since, further, an arc which contains points within triangles of $[t]_1$ and also within $[t]_2$ must contain points exterior to every triangle of both sets, it follows that only a finite number of the set $[a]_i$ ($i \gtreqless k$) can fail to contain points exterior to both sets of triangles. Denote by $[Q]$ the set of all points of $[a]_i$ ($i \gtreqless k$) which are exterior to the triangles of both sets, together with the limit-points of such points. Then $[A]_1$, $[A]_2$ and $[Q]$ are closed sets, no one containing a limit-point of the others. Hence we can place a finite set of triangles about the set $[Q]$ without enclosing any point of $[A]_1$ or $[A]_2$. Since an infinitude of arcs of every set $[a]_i$, ($i \gtreqless k$) contains points of $Q$, it follows by the definition of $P_{[a]_i, j}$ that for some value of $i$, as $i=h$, there will be a point of $P_{[a]_i, j}$ on every arc of $[a]_i$ which contains a point in $[Q]$, and hence there will be a limit-point of such points, that is, a point of $A_{i,j}$ which does not lie within a triangle of either of the sets $[t]_1$ and $[t]_2$, which is contrary to the assumption that all points of $[A]$ are points of one or the other of the sets $[A]_1$ and $[A]_2$.

(c) *The set $[A]$ contains no connected subset containing $A$ and $B$.*

We show first that if a point $A^{(1)}$ of $A_{i,j}$ other than $A$ or $B$ is removed from $[A]$ the remaining set is not connected. This is done by showing that the set of points of $A_{i,j}$ between $A$ and $A^{(1)}$ have no limit-point other than $A^{(1)}$ in common with the points of $A_{i,j}$ between $A^{(1)}$ and $B$. Suppose there is such common limit-point $Q$. About $Q$ set a triangle $t_1$ of which $A^{(1)}$ is an exterior point, and about $A^{(1)}$ set a triangle $t_2$ of which no point of $t_1$ is an interior point. About $Q$ set another triangle $t_3$ such that by the uniform continuity of the curves of $[a]$ with respect to the set no curve contains a point of $t_1$ between points of $t_3$. Since $A^{(1)}$ is a limit-point of a certain set of points $P_{[a]_i,j}$ (for fixed $j$), it follows that there must be some arc which contains interior points of $t_2$ between points of $t_3$, and which therefore contains points of $t_1$ between points of $t_3$, which contradicts the properties assumed for $t_1$ and $t_3$. Hence the subset remaining when a point of $A_{i,j}$ is removed from $[A]$ is not connected. Consider now any point $\bar{A}$ of $[A]$ but not of $A_{i,j}$. By the preceding, the point $\bar{A}$ distinguishes the points of $A_{i,j}$ into two classes $[A]'$ and $[A]''$ such that $\bar{A}$ is a limit-point of points of $A_{i,j}$ between any point of $[A]'$ and the point $B$, and also of such points between any point of $[A]''$ and $A$. Now if possible let $Q$ be a common limit-point of $[A]'$ and $[A]''$ other than $\bar{A}$. As above, set triangles about $\bar{A}$ and $Q$, neither containing as interior point a point of the other, when the argument to show that $Q$ is not a limit-point of both $[A]'$ and $[A]''$ is like that given above.

2. THEOREM. *For every entirely open set $R$ containing all points of $[A]$ there is a value of $i$, $i=k$, such that for all values of $i \geqq k$ only a finite number of arcs of $[a]_i$ fail to lie entirely within $R$.*

PROOF. Set about the points of $[A]$ a finite number of triangles $[t]_1$ every interior point of which is a point of $R$. About a point $A_1$ of $[A]$, within a triangle $t_1$ of $[t]_1$, set a triangle $t^{(1)}$ within $t_1$ such that no arc of $[a]$ contains points on $t_1$ between points on $t^{(1)}$; and similarly for every point of $[A]$. Then we obtain a finite subset $[t]'$ of $m$ of these triangles enclosing all points of $[A]$. These triangles may be ordered so that $A$ lies within $t^{(1)}$ and $B$ lies within $t^{(m)}$, and such that $t^{(i)}$ has interior points in common with both $t^{(i-1)}$ and $t^{(i+1)}$ $(i=2, \ldots, m-1)$. Let $A^{(1)}$ be a point of $A_{i,j}$ within both $t^{(1)}$ and $t^{(2)}$, and in general $A^{(i)}$ a point of $A_{i,j}$ within both $t^{(i)}$ and $t^{(i+1)}$. Then there is a value of $i$, $\bar{\imath}$, for which all points $A^{(i)}$ $(i=1, \ldots, m)$ are limit-points of $P_{[a]_{\bar{\imath}},j}$. Then only a finite number of arcs of the set $[a]_{\bar{\imath}}$ contain points exterior to $[t]_1$.

DEFINITION. *The arc specified in the preceding theorem is said to be a limit-arc of the set, and the type of approach specified in* (2) *is called uniform approach.*

We now summarize the preceding in the following theorem:

3. THEOREM. *If* $[a]$ *is a set of simple continuous arcs connecting two points* $A$ *and* $B$ *and lying in a closed region* $R$, *and if the arcs are uniformly continuous with respect to the whole set, then there is a continuous arc in* $R$ *connecting* $A$ *and* $B$ *which is a limit-arc of the set* $[a]$ *and which is approached uniformly by a certain subset of* $[a]$.

## § 8. *Concerning the Existence of Minimizing Curves.*

(We now use the usual metric hypotheses of geometry and the Cartesian correspondence between points in a plane and the pairs of real numbers.) Consider a function $f(x, y)$ defined over a closed region $R$, and continuous and positive in that region. Let $a_1$ be an arc of finite length lying in $R$ and connecting two of its points $A$ and $B$. Let $A, P_1, \ldots, P_i, \ldots, P_n = B$ be a set of $n$ points on $a_1$ lying in order on it from $A$ to $B$, as indicated by the notation. Denote by $[\sigma_i]$ the length of the set of chords $AP_1, P_1P_2, \ldots, P_{i-1}P_i$; and let $\xi_i$ be the values of $f(x, y)$ at the points $P_i$. Denote by $\Delta$ the length of the longest segment of $[\sigma_i]$. Then

$$a_1 \int_A^B f(x, y) = \underset{\Delta \doteq 0}{L} \Sigma \sigma_i \cdot \xi_i \qquad (i = 1, \ldots, n).$$

This limit will necessarily exist and be finite if $a_1$ is continuous and of finite length and $f(x, y)$ is continuous.

In this manner a definite positive number $N(a)$ is associated with every arc $a$ of finite length lying in $R$ and connecting $A$ and $B$. Consider now the set $[a]$ of all continuous arcs of finite length lying in $R$ and connecting $A$ and $B$. The lower bound $\underline{B}$ of the set of numbers $[N(a)]$ is greater than zero; *i. e.*, it is greater than or equal to the distance from $A$ to $B$ multiplied by the minimum of $f(x, y)$ in $R$. Select an infinite sequence of arcs $\{a_i\}$ such that the sequence of numbers $\{N(a_i)\}$ is non-oscillating decreasing with the limit $\underline{B}$.

1. THEOREM. *The arcs of the sequence* $\{a_i\}$ *satisfy the condition of uniform continuity with respect to the set of arcs* $\{a_i\}$ (*see* (3), §7), *and hence have at least one limit-curve.*

PROOF. Consider any point $P$ of $R$. If there exists a neighborhood of $P$ within which are points of only a finite number of arcs of $\{a_i\}$, then the condition is a direct consequence of the continuity of each arc. If there is an infinitude of arcs of $\{a_i\}$ containing points within every neighborhood of $P$, set any

triangle $t_1$ about $P$. Let the shortest distance from $P$ to $t_1$ be $d_1$, and let $M$ and $m$ be the maximum and minimum respectively of $f(x, y)$ in $R$. Set about $P$ a triangle $t_2$ with shortest distance $d_2$ from $P$ to $t_2$ such that $d_2 M < \frac{d_1 m}{2}$.

Since $\underset{i \doteq \infty}{L} \{N(a_i)\} = \underline{B}$, it follows that for some value of $i$, as $i_1$, every arc $a_{i_1+k}$ ($k = 0, \ldots, \infty$) must fail to contain points of $t_1$ between points of $t_2$. Any triangle $t_3$ within $t_2$ and enclosing $P$ which satisfies the condition for the finite set of arcs $a_1, \ldots, a_{i_1}$ must therefore satisfy the condition for the whole sequence $\{a_i\}$. Hence, by (3), § 7, there is a limit-arc $\bar{a}$ of the sequence $\{a_i\}$.

2. THEOREM. *The lengths of the arcs $\{a_i\}$ have a finite upper bound, and their limit-arc $\bar{a}$ is finite in length.*

PROOF. If the lengths of the arcs formed an unbounded set, the integrals would be an unbounded set, inasmuch as the integral of each curve is equal to or greater than its length multiplied by $m$. Let $\overline{B}$ be the upper bound of the lengths of the arcs of $\{a_i\}$. If $\bar{a}$ is of infinite length, we can find a set of points $P_1, P_i, \ldots, P_n$ on it such that $\sum_{i=1}^{i=n} \sigma_i$ shall be greater than $\overline{B} + d$, where $d$ is any preassigned number. About each point $P_i$ set a circle $c_i$ of radius $r$, where $r < \frac{d}{2n}$ ($n$ being the number of points $P_i$) or $2nr < d$. By (3), § 7, there is an arc $a^{(k)}$ of $\{a_i\}$ which contains points within every circle $c_1$. Hence the arc $a^{(k)}$ can not be less than $\Sigma \sigma_i$ by more than $2nr$. That is, it is greater than $\overline{B}$, which is contrary to the definition of $\overline{B}$.

DEFINITION. *Any number of a set such that there is no number of the set less than it, is called an "absolute minimum" of the set. A curve $\bar{a}$ of a set $[a]$, such that $N(\bar{a})$ is an absolute minimum of the set of numbers $[N(a)]$, is called a minimizing curve of the set.*

3. THEOREM. *If $f(x, y)$ is a continuous positive function defined over a closed, bounded set of points $R$, and if there exists an arc of finite length lying in $R$ and connecting $A$ and $B$, then there exists at least one such arc $\bar{a}$ in $R$ for which $N(\bar{a})$ is an absolute minimum.*

PROOF. Suppose $\bar{a}$ is not a minimizing curve. Then there is some fixed positive number $d$ such that, within every entirely open set containing $\bar{a}$, there is an arc $a^{(k)}$ such that $N(\bar{a}) - N(a^{(k)}) > d$. Let $A = P_1, \ldots, P_i, \ldots, P_n = B$ be a set of points $P_i$ on $\bar{a}$, and $\sigma_i$ the length of the corresponding chords such that

$$|\Sigma \sigma_i \cdot \xi_i - N(\bar{a})| < \frac{d}{4}.$$

By a well-known property of continuous curves, there exists a positive number $r$, such that, if circles $c_i$ of radius $r$ are described about $P_i$ as centers, there are no points of the arcs $A P_{i-1}$ and $P_{i+1} A$ within the circle $c_i$, and, further, such that each circle is entirely exterior to every other. Within each circle $c_i$ place a concentric circle $c_i^*$ with radius $\frac{r}{2}$. From $P_i$ trace the arc $\bar{a}$ towards $A$ until first meeting $c_i^*$ in a point $P_i'$. Similarly trace $\bar{a}$ from $P_i$ towards $B$ until first meeting $c_i^*$ in $P_i''$. Consider the resulting set of arcs $A P_1', \ldots,$ $P_{i-1}'' P_i'$, $P_i'' P_{i+1}', \ldots$ (leaving out the arc $P_i' P_i''$). Between any two of these arcs there is a minimum distance. Let $d_1$ be the smallest of these distances. Using a circle whose radius $r_1$ is less than $\frac{d_1}{2}$, trace a neighborhood of $\bar{a}$ by letting the center of the circle pass over $\bar{a}$ from $A$ to B. Denote this neighborhood of $\bar{a}$ by $R^{(1)}$. The circles $c_i$ divide that part of $R_1$ which is exterior to them into a set of $n$ regions $R_i$ lying in order about $\bar{a}$ from $A$ to $B$. In any one of these regions $R_i$ there is a definite difference between the maximum and minimum values of $f(x, y)$. Denote by $v_i$ the difference between the maximum and minimum values of $f(x, y)$ in $R_{i-1}$, $R_i$ and $R_{i+1}$ and the two circles $c_i$ and $c_{i+1}$, and let $V$ be the greatest of these values of $v_i$. Denote by $l$ the length of $\bar{a}$ and by $l_i$ the length of the arcs $P_{i-1} P_i$.

Let $a^{(k)}$ be any arc of $\{a_i\}$ within $R^{(1)}$. Then $a^{(k)}$ must contain points within each circle $c_i$. Construct a broken line $[\sigma_j^{(k)}]$ with vertices on $a^{(k)}$, having the following properties:

(a) $|\Sigma \sigma_j^{(k)} \cdot \xi_j^{(k)} - N(a^{(k)})| < \frac{d}{4}$.

(b) There is a vertex $P_i^{(k)}$ of $\sigma_j^{(k)}$ within each circle $c_i$.

Denote by $l_i^{(k)}$ the length of the broken line of $\sigma_j^{(k)}$ connecting $P_{i-1}^{(k)}$ and $P_i^{(k)}$. We note that $\sigma_i \gtreqless l_i$. Then $\sigma_i$ can not exceed $l_i^{(k)}$ by more than $4r$, and hence $\sigma_i \cdot \xi_i$ can not exceed $\sigma_j^{(k)} \cdot \xi_j^{(k)}$ taken from $P_{i-1}^{(k)}$ to $P_i^{(k)}$ by more than $l_i V + 4rM$, where $M$ is the maximum of $f(x, y)$ in $R$. Hence $\Sigma \sigma_i \cdot \xi_i$ can not exceed $\Sigma \sigma_j^{(k)} \cdot \xi_j^{(k)}$ by more than

$$V \Sigma l_i + 4 n r M \quad \text{or} \quad V l + 4 n r M.$$

In this expression $l$ and $M$ are fixed. $V$ and $n$ are fixed simultaneously and $r$ is fixed independently of $V$ and $n$. The processes are as follows: First impose on the points $P_i$ of $\bar{a}$ the further condition that the maximum of the differences

between the maximum and minimum of $f(x, y)$, on an arc consisting of any three of the consecutive arcs into which $P_i$ divides $\bar{a}$, shall be $V' < \frac{d}{8l}$. Then we can impose upon $r_1$ (the radius of the circle which traces out the region $R_1$) the additional condition that the variation $V$ described above shall not be greater than $2V'$, whence $V < \frac{d}{4l}$ or $Vl < \frac{d}{4}$. We now impose upon $r$ the further condition that $r < \frac{d}{16nM}$ (note that $n$ is fixed when the points $P_i$ are determined) or $4nrM < \frac{d}{4}$. Then

$$Vl + 4nrM < \frac{d}{2}.$$

Since, therefore, $\Sigma\alpha_i \cdot \xi_i$ can not exceed $\Sigma\,\sigma_j^{(k)} \cdot \xi_j^{(k)}$ by $\frac{d}{2}$, and since

$$|\Sigma\,\sigma_j^{(k)} \cdot \xi_j^{(k)} - N(a^{(k)})| < \frac{d}{4},$$

$$|\Sigma\,\sigma_i \cdot \xi_i - N(\bar{a})| < \frac{d}{4},$$

it follows that $N(\bar{a})$ can not exceed $N(a^{(k)})$ by $d$, which proves our theorem.

This theorem covers a special case of the general problem of existence of solutions in the calculus of variations. There the problem is to minimize the integral $\int f(x, y, y')\,dx$, whereas in the present theorem $y'$ is not present. However, all cases where the expression to be minimized can be written in the form $\int f(x, y)\sqrt{1 + y'^2}\,dx$ come under the case treated here; *i. e.*, where the $y'$ enter simply to involve the length of arc as a multiplicative factor. Thus the existence of a *shortest distance* between any two points in a closed connected region (if any arc of finite length in the region connects them) and the existence of a minimum surface of revolution follow from this theorem. It should be noted that no assumption, other than that of finite length, as to the character of the curves is made. The theorem says that among *all* curves there is *a curve* which has the required property. Further, there is no assumption as to the character of the boundary of the region in which the curves lie. For the purpose of this discussion region may be any connected set whatever.

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